MSO202A: Assignment-III

1. Determine all $z \in \mathbb{C}$ for which the following series converge absolutely.

$$(a)\ \sum \frac{z^n}{n^2} \qquad (b)\ \sum \frac{z^n}{n!} \qquad (c)\ \sum \frac{1}{n!}\frac{1}{z^n} \qquad (d)\ \sum \frac{1}{2^n}\frac{1}{z^n}$$

2. Let $a_n = \frac{(-1)^n}{\sqrt{n}} + i\frac{1}{n^2}$ for $n = 1, 2, 3, \cdots$. Show that the series $\sum a_n$ converges but it does not converge absolutely.

3. The following series $\sum z^n$, $\sum z^n/n$ and $\sum z^n/n^2$ have radius of convergence 1. Show that the series
   (a) $\sum z^n$ does not converge for any z such that $|z| = 1$,
   (b) $\sum z^n/n$ converges for all z for which $z \neq 1$ and $|z| = 1$ and
   (c) $\sum z^n/n^2$ converges for all z such that $|z| = 1$.

4. Find the radius of convergence of the power series $\sum a_n(z-a)^n$ for which
   (a) $a_n = r^n/n^p$ where r and p are two positive real numbers
   (b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$
   (c) $a_n = \frac{1}{2^n-1}$

5. Find the radius of convergence of the following power series
   (a) $\sum 2nz^n$
   (b) $\sum n!z^{2n+1}$
   (c) $\sum(-1)^n\frac{z^{2n}}{(2n)!}$

6. If R_1 and R_2 are the radii of convergence of the series $\sum a_nz^n$ and $\sum b_nz^n$ respectively, then show that $R \geq \min\{R_1, R_2\}$ is the radius of convergence of the series $\sum(a_n + b_n)z^n$.

7. Show that $\sum_{n=0}^{\infty}(n+1)^2z^n = \frac{1+z}{(1-z)^3}$ for $|z| < 1$.

8. Find i^i and $\cosh(\operatorname{Log} 4)$. (Log stands for the principal branch of the logarithm)

9. For $z_1, z_2 \in G = \{re^{i\theta} : r > 0, -\pi < \theta < \pi\}$, is it always true that $\operatorname{Log}(z_1z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$? Find the conditions on z_1 and z_2 so that the equality holds.

10. Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$. Hence prove that cos function is not bounded in $\mathbb{C}$. Also, find the zeros of $\cos z$.

11. Show that $\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$.

12. Show that $\sin \bar{z}$ and $\cos \bar{z}$ are not analytic functions on any domain.

13. Find all solutions z of (a) $\cos z = 2$ (b) $\sin\theta \sin z = 1$ where $\theta \in \mathbb{R}$ (c) $|\cot z| = 1$

14. Express in the form $a + ib$: (a) $\log \operatorname{Log}$ i (b) $(-3)^{\sqrt{2}}$ (c) i^{-i}

15. Show that (a) $\sin^{-1} z = -i\log(iz + \sqrt{1-z^2})$ (b) $\cot^{-1} z = \frac{i}{2}\log(z-i)/(z+i)$ (c) $\cosh^{-1} z = \log(z + \sqrt{z^2-1})$