MSO202A: Assignment-IV

1. Evaluate

   (a) $\int_C |z|\frac{z}{\bar{z}}\, dz$ where C is the clockwise oriented boundary of the part of the annulus $2 \le |z| \le 4$ lying in the third and fourth quadrants.

   (b) $\int_C \frac{1}{\sqrt{z}} dz$ where C is the counterclockwise oriented semicircular part of the circle $|z| = 1$ in the lower half plane and $\sqrt{z}$ is defined such that $\sqrt{1} = -1$.

   (c) $\int_C (z-a)^m dz$, where $m \in \mathbb{Z}$ and C is the semicircle $|z-a| = r$, $0 \le \arg(z-a) \le \pi$

   (d) $\int_C (z-a)^m dz$, where $m \in \mathbb{Z}$ and C is the circle $|z-a| = r$, $0 \le \arg(z-a) \le 2\pi$

2. Without actually evaluating the integral, prove that

   (a) $|\int_\gamma \frac{dz}{z^2-1}| \le \pi/3$, where $\gamma(t) = 2e^{it}$ for $0 \le t \le \pi/2$.

   (b) $|\int_C \frac{dz}{z^2+1}| \le 2\pi/(3-2\sqrt{2})$, where C is the circle $|z-1| = 1$.

3. Let γ_1 be a semicircular path joining -1 and 1 with centre at 0 and γ_2 a rectangular path with vertices -1, $-1+i$, $1+i$ and 1. Find $\int_{\gamma_1} \bar{z}\, dz$ and $\int_{\gamma_2} \bar{z}\, dz$ (observe path dependence).

4. Evaluate

$$(a)\ \int_{|z|=2} \frac{z}{z^2-1} dz \quad (b)\ \int_{|z|=2} \frac{z}{(z^2-1)^2} dz \quad (c)\ \int_{|z|=2} \frac{e^{2z}}{z(z+1)^4} dz$$

5. Show that $\int_\gamma \frac{e^z}{z} dz = 2\pi i$, where $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$. Using this, evaluate

$$(a)\ \int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta \qquad (b)\ \int_0^{2\pi} e^{k\cos\theta} \sin(k\sin\theta)\, d\theta$$

6. Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$. Find $\int_C P(z)/z^k\, dz$ where $C : |z| = R$ and $k \in \mathbb{N} \cup \{0\}$.

7. Let $C : |z| = 2$. Find the values of $\int_C z^n (1-z)^m\, dz$ for $m \in \mathbb{N} \cup \{0\}, n \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z}$

8. Evaluate the integral $\int_C \frac{dz}{z(z^2+1)}$ for all possible choice of the closed contour C that does not pass through $0, i, -i$.

9. Show that $\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi x\xi} dx = e^{-\pi\xi^2}$ for $\xi \in \mathbb{R}$ by integrating $f(z) = e^{-z^2}$ along the lines of a rectangle with vertices $R, R+i\xi, -R+i\xi, -R$

10. Show that $\int_{|z|=2} \frac{e^{az}}{z^2+1} dz = 2\pi i \sin a$

11. Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\{z \in \mathbb{C} : z \ne 0\}$ and bounded on the set $\{z \in \mathbb{C} : |z| \le 1/2\}$. Prove that $\int_{|z|=R} f(z) dz = 0$ for every $R > 0$.

12. Show that $\left|\int_{|z|=R} \frac{\operatorname{Log} z}{z^2} dz\right| \leq 2\sqrt{2}\pi \frac{\ln R}{R}$, $R > e^{\pi}$.

13. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function where $\mathbb{D}$ is the open unit disk. If $d = \sup_{z,w\in\mathbb{D}} |f(z)- f(w)|$, then show that $2|f'(0)| \leq d$.

14. Prove Mean Value Theorem: Let Ω be an open set and $f : \Omega \to \mathbb{C}$ be an analytic function. Then $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$ for every $r > 0$ such that the open ball $B(z_0, r)$ is contained in Ω. Further show that if $f(z_0) = 0$ for some $z_0 \in \Omega$, then $\mathrm{Re}(f)$ takes both positive and negative values on the circle which is the boundary of $B(z_0, r)$ for every $r > 0$.

15. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leq A + B|z|^k$ for some $k \in \mathbb{N}$ where $A > 0, B > 0$. Show that f is a polynomial of degree at most k.

16. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 0$. Show that f is constant.

17. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that the image of the function has to necessarily meet the real axis and imaginary axis.

18. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function such that $f(0) = 0$. Show that (a) $|f(z)| \leq |z|$ for all $z \in \mathbb{C}$ and $|f'(0)| \leq 1$, (b) If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}$ or $|f'(0)| = 1$, then there exists $c \in \mathbb{C}$ such that $|c| = 1$ and $f(z) = cz$ for all $z \in \mathbb{D}$.

19. Let $f_j : \mathbb{C} \to \mathbb{C}$, $j = 1, 2$ be analytic functions such that $f_1(a_n) = f_2(a_n)$ for a bounded sequence of complex numbers. Show that the functions are same.

20. Find the maximum of the function $|f|$ on $\overline{\mathbb{D}}$ (closed unit disk) for (a) $f(z) = z^2 - z$ and (b) $f(z) = \sin z$.