MTH203: Assignment-7

1.D Expand the following functions in terms of Legendre polynomials over $[-1, 1]$:

(i) $f(x) = x^3 + x + 1$ (ii) $f(x) = \begin{cases} 0 & \text{if} \quad -1 \le x < 0 \\ x & \text{if} \quad 0 \le x \le 1 \end{cases}$ (first three nonzero terms)

2.T Locate and classify the singular points in the following:
(i) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$ (ii) $(3x+1)xy'' - xy' + 2y = 0$

3.T For each of the following, verify that the origin is a regular singular point and find two linearly independent solutions:
(a) $9x^2y'' + (9x^2+2)y = 0$ (b) $x^2(x^2-1)y'' - x(1+x^2)y' + (1+x^2)y = 0$
(c) $xy'' + (1-2x)y' + (x-1)y = 0$ (d) $x(x-1)y'' + 2(2x-1)y' + 2y = 0$

4.D Show that $2x^3y'' + (\cos 2x - 1)y' + 2xy = 0$ has only one Frobenius series solution.

5.D Reduce $x^2y'' + xy' + (x^2 - 1/4)y = 0$ to normal form and hence find its general solution.

6.D Find a solution bounded near $x = 0$ of the following ODE

$$x^2y'' + xy' + (\lambda^2x^2 - 1)y = 0$$

7.D Using recurrence relations, show that
(i) $J_0''(x) = -J_0(x) + J_1(x)/x$ (ii) $xJ_{n+1}'(x) + (n+1)J_{n+1}(x) = xJ_n(x)$

8.D Show that
(i) $\int x^4 J_1(x)\,dx = (4x^3 - 16x)J_1(x) - (x^4 - 8x^2)J_0(x) + C$
(ii) $\int J_5(x)\,dx = -2J_4(x) - 2J_2(x) - J_0(x) + C$

9.D Express
(i) $J_3(x)$ in terms of $J_1(x)$ and $J_0(x)$ (ii) $J_2'(x)$ in terms of $J_1(x)$ and $J_0(x)$
(iii) $J_4(ax)$ in terms of $J_1(ax)$ and $J_0(ax)$

10.D Prove that between each pair of consecutive positive zeros of $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$ and vice versa.

11.T Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$ on a closed interval $[a, b]$. Show that $u(x)$ has at most a finite number of zeros in $[a, b]$.

12.T Show that any nontrivial solution of $u'' + q(x)u = 0$, $q(x) < 0$ has at most one zero.

13.D Let $u(x)$ be any nontrivial solution of $u'' + [1 + q(x)]u = 0$, where $q(x) > 0$. Show that $u(x)$ has infinitely many zeros.

14.T Let y_ν be a nontrivial solution of Bessel's equation of order ν on the positive x-axis. Show that (i) if $0 \le \nu < 1/2$, then every interval of length π contains at least one zero of $y_\nu(x)$; (ii) if $\nu = 1/2$, then the distance between successive zeros of y_ν is exactly π; and (iii) if $\nu > 1/2$, then every interval of length π contains at most one zero of $y_\nu(x)$.

15.T Show that the Bessel functions J_ν ($\nu \ge 0$) satisfy

$$\int_0^1 xJ_\nu(\lambda_m x)J_\nu(\lambda_n x)\,dx = \frac{1}{2}J_{\nu+1}^2(\lambda_n)\delta_{mn},$$

where λ_i are the positive zeros of J_ν.