Lecture XI
Euler-Cauchy Equation

1 Homogeneous Euler-Cauchy equation

If the ODE is of the form

\[ ax^2 y'' + bxy'' + cy = 0, \quad (1) \]

where \( a, b \) and \( c \) are constants; then (1) is called homogeneous Euler-Cauchy equation. Two linearly independent solutions (i.e. basis) depend on the quadratic equation

\[ am^2 + (b - a)m + c = 0. \quad (2) \]

Equation (2) is called characteristic equation for (1). The ODE (1) is singular at \( x = 0 \). Hence, we solve (1) for \( x \neq 0 \). We consider the case when \( x > 0 \).

**Theorem 1.** (i) If the roots of (2) are real and distinct, say \( m_1 \) and \( m_2 \), then two linearly independent (LI) solutions of (1) are \( x^{m_1} \) and \( x^{m_2} \). Thus, the general solution to (1) is

\[ y = C_1 x^{m_1} + C_2 x^{m_2}. \]

(ii) If the roots of (2) are real and equal, say \( m_1 = m_2 = m \), then two LI solutions of (1) are \( x^m \) and \( x^m \ln x \). Thus, the general solution to (1) is

\[ y = (C_1 + C_2 \ln x)x^m. \]

(iii) If the roots of (2) are complex conjugate, say \( m_1 = \alpha + i\beta \) and \( m_2 = \alpha - i\beta \), then two real LI solutions of (1) are \( x^\alpha \cos(\beta \ln x) \) and \( x^\alpha \sin(\beta \ln x) \). Thus, the general solution to (1) is

\[ y = x^\alpha \left( C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x) \right). \]

**Proof:** We have seen that the trial solution for a constant coefficient equation is \( e^{mx} \). Now since power of \( x^m \) is reduced by 1 by a differentiation, let us take \( x^m \) as trial solution for (1).

For convenience, (1) is written in the operator form \( L(y) = 0 \), where

\[ L \equiv ax^2 \frac{d^2}{dx^2} + bx \frac{d}{dx} + c. \]

We also sometimes write \( L \) as

\[ L \equiv ax^2 D^2 + bx D + c, \]

where \( D = d/dx \). Now

\[ L(x^m) = \left( am(m - 1) + bm + c \right) x^m = p(m) x^m, \quad (3) \]

where \( p(m) = am^2 + (b - a)m + c \). Thus, \( x^m \) is a solution of (1) if \( p(m) = 0 \).

(i) If \( p(m) = 0 \) has two distinct real roots \( m_1, m_2 \), then both \( x^{m_1} \) and \( x^{m_2} \) are solutions of (1). Since, \( m_1 \neq m_2 \), they are also LI. Thus, the general solution to (1) is

\[ y = C_1 x^{m_1} + C_2 x^{m_2}. \]
Example 1. Solve \( x^2y'' - xy' - 3y = 0 \)

Solution: The characteristic equation is \( m^2 - 2m - 3 = 0 \) \( \Rightarrow m = -1, 3 \). The general solution is \( y = C_1/x + C_2x^3 \).

(ii) If \( p(m) = 0 \) has real equal roots \( m_1 = m_2 = m \), then \( x^m \) is a solution of (1). To find the other solution, note that if \( m \) is repeated root, then \( p(m) = p'(m) = 0 \). This suggests differentiating (3) w.r.t. \( m \). Since \( L \) consists of differentiation w.r.t. \( x \) only,

\[
\frac{\partial}{\partial m} \left( L(x^m) \right) = L \left( \frac{\partial}{\partial m} x^m \right) = L(x^m \ln x).
\]

Now

\[
L(x^m \ln x) = (p'(m) + p(m) \ln x)x^m,
\]

where \( ' \) represents the derivative. Since, \( m \) is a repeated root, the RHS is zero. Thus, \( x^m \ln x \) is also a solution to (1) and it is independent of \( x^m \). Hence, the general solution to (1) is

\[
y = (C_1 + C_2 \ln x)x^m.
\]

(We can also use method of reduction of order technique i.e. \( y_1 = x^m \) and \( y_2 = v(x)y_1 = v(x)x^m \). From the given ODE, we find

\[
ax^2v'' + (2am + b) xv' + \left( am^2 + (b - a)m + c \right) v = 0
\]

Since \( m = m_1 = m_2 \) is a double root, we must have \( am^2 + (b - a)m + c = 0 \) and \( m = -(b - a)/2a \Rightarrow 2am + b = a \). Hence,

\[
ax^2v'' + av' = 0 \Rightarrow v'' = -\frac{v'}{x} \Rightarrow v' = \frac{1}{x} \Rightarrow v = \ln x
\]

Hence \( y_2 = x^m \ln x \)

Example 2. Solve \( x^2y'' - 3xy' + 4y = 0 \)

Solution: The characteristic equation is \( m^2 - 4m + 4 = 0 \) \( \Rightarrow m = 2, 2 \). The general solution is \( y = (C_1 + C_2 \ln x)x^2 \).

(iii) If \( p(m) = 0 \) has complex conjugate roots, say \( m_1 = \alpha + i\beta \) and \( m_2 = \alpha - i\beta \), then two LI solutions are

\[
Y_1 = x^{(\alpha+i\beta)} = x^\alpha e^{i\beta \ln x}, \quad \text{and} \quad Y_2 = x^\alpha e^{-i\beta \ln x}.
\]

But these are complex valued. Note that if \( Y_1, Y_2 \) are LI, then so are \( y_1 = (Y_1 + Y_2)/2 \) and \( y_2 = (Y_1 - Y_2)/2i \). Hence, two real LI solutions of (1) are \( y_1 = x^\alpha \cos(\beta \ln x) \) and \( y_2 = x^\alpha \sin(\beta \ln x) \). Thus, the general solution to (1) is

\[
y = x^\alpha \left( C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x) \right).
\]

Example 3. Solve \( x^2y'' - 3xy' + 5y = 0 \)
Solution: The characteristic equation is \( m^2 - 4m + 5 = 0 \) \( \Rightarrow m = 2 \pm i \). The general solution is \( y = x^2 \left( C_1 \cos(\ln x) + C_2 \sin(\ln x) \right) \)

Comment 1: The solution for \( x < 0 \) can be obtained from that of \( x > 0 \) by replacing \( x \) by \(-x\) everywhere.

Comment 2: Homogeneous Euler-Cauchy equation can be transformed to linear constant coefficient homogeneous equation by changing the independent variable to \( t = \ln x \) for \( x > 0 \).

Comment 3: This can be generalized to equations of the form

\[
a(\gamma x + \delta)^2y'' + b(\gamma x + \delta)y' + cy = 0.
\]

In this case we consider \((\gamma x + \delta)^m\) as the trial solution.

2 Nonhomogeneous Euler-Cauchy equation

If the ODE is of the form

\[
a x^2 y'' + b x y' + c y = \tilde{r}(x),
\]

where \( a, b \) and \( c \) are constants; then (4) is called nonhomogeneous Euler-Cauchy equation. We can use the method of variation of parameters as follows. First divide (4) by \( a x^2 \) so that the coefficient of \( y'' \) becomes unity:

\[
y'' + \frac{b}{a x} y' + \frac{c}{a x^2} y = r(x),
\]

where \( r(x) = \tilde{r}(x)/a x^2 \). Now we already know two LI solutions \( y_1, y_2 \) of the homogeneous part. Hence, the particular solution to (4) is given by

\[
y_p(x) = -y_1(x) \int \frac{y_2(x) r(x)}{W(y_1, y_2)} dx + y_2(x) \int \frac{y_1(x) r(x)}{W(y_1, y_2)} dx.
\]

Thus, the general solution to (4) is

\[
y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x).
\]

Comment: In few cases, it can be solved also using method of undetermined coefficients. For this, we first convert it to constant coefficient liner ODE by \( t = \ln x \). If the the transformed RHS is of special form then the method of undetermined coefficients is applicable.

Example 4. Consider

\[
x^2 y'' - x y' - 3y = \frac{\ln x}{x}, \quad x > 0.
\]

The characteristic equation is \( m^2 - 2m - 3 = 0 \) \( \Rightarrow m = -1, 3 \). Hence \( y_1 = 1/x \) and \( y_2 = x^3 \). Hence,

\[
y_p(x) = y_1(x) u(x) + y_2(x) v(x)
\]
where

\[ u(x) = - \int \frac{y_2(x)r(x)}{W(y_1, y_2)} \, dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)} \, dx. \]

Now \( W(y_1, y_2) = 4x \) and \( r(x) = \ln x/x^3 \). Hence,

\[
\begin{align*}
    u(x) &= - \int \frac{\ln x}{4x} \, dx = - \frac{(\ln x)^2}{8} \\
    v(x) &= \int \frac{\ln x}{4x^5} \, dx = - \frac{\ln x}{16x^4} - \frac{1}{64x^4}
\end{align*}
\]

Hence,

\[
y_p(x) = - \frac{(\ln x)^2}{8x} - \frac{\ln x}{16x} - \frac{1}{64x}
\]

Hence the general solution is \( y = c_1y_1 + c_2y_2 + y_p \), i.e.

\[
y(x) = \frac{A}{x} + Bx^3 - \frac{(\ln x)^2}{8x} - \frac{\ln x}{16x}.
\]

Note that last term of \( y_p \) is absorbed with \( y_1 \).

**Aliter**: Let us make the transformation \( t = \ln x \). Let \( y(x) = y(e^t) = u(t) \). Then the given transformed to

\[
\ddot{u} - 2\dot{u} - 3u = te^{-t},
\]

where \( \dot{\cdot} = d/dt \). This is the same problem we have solved in lecture 9 using method of undetermined coefficients. The solution is (see lecture 9)

\[
u(t) = C_1e^{-t} + C_2e^{3t} - \frac{te^{-t}}{16}(2t + 1) = y(e^t),
\]

which in terms of original \( x \) variable becomes

\[
y(x) = \frac{C_1}{x} + C_2x^3 - \frac{\ln x}{16x}(2\ln x + 1),
\]