## Lecture XIV

Frobenius series: Regular singular points

## 1 Singular points

Consider the second order linear homogeneous equation

$$
\begin{equation*}
a_{0}(x) y^{\prime \prime}+a_{1}(x) y^{\prime}+a_{2}(x) y=0, \quad x \in \mathcal{I} \tag{1}
\end{equation*}
$$

Suppose that $a_{0}, a_{1}$ and $a_{2}$ are analytic at $x_{0} \in \mathcal{I}$. If $a_{0}\left(x_{0}\right)=0$, then $x_{0}$ is a singular point for (1).

Definition 1. A point $x_{0} \in \mathcal{I}$ is a regular singular point for (1) if (1) can be written as

$$
\begin{equation*}
b_{0}(x)\left(x-x_{0}\right)^{2} y^{\prime \prime}+b_{1}(x)\left(x-x_{0}\right) y^{\prime}+b_{2}(x) y=0 \tag{2}
\end{equation*}
$$

where $b_{0}\left(x_{0}\right) \neq 0$ and $b_{0}, b_{1}, b_{2}$ are analytic at $x_{0}$.
Comment 1: Since $b_{0}\left(x_{0}\right) \neq 0$, we get an equivalent definition of regular singular point by dividing (2) by $b_{0}(x)$. Thus, a point $x_{0} \in \mathcal{I}$ is a regular singular point for (1) if (1) can be written as

$$
\begin{equation*}
\left(x-x_{0}\right)^{2} y^{\prime \prime}+\left(x-x_{0}\right) p(x) y^{\prime}+q(x) y=0, \tag{3}
\end{equation*}
$$

where $p$ and $q$ are analytic at $x_{0}$.
Comment 2: Any singular point of (1) which is not regular is called irregular singular point.

Example 1. Consider

$$
x^{3} y^{\prime \prime}-(1-\cos x) y^{\prime}+x y=0
$$

The singular point $x_{0}=0$ is regular.
Example 2. Consider

$$
x^{2}(x-1)^{2} y^{\prime \prime}+(\sin x) y^{\prime}+(x-1) y=0
$$

The singular point $x_{0}=0$ is regular whereas $x_{0}=1$ is irregular.
Example 3. Euler-Cauchy equation:

$$
a x^{2} y^{\prime \prime}+b x y^{\prime}+c y=0,
$$

where $a, b, c$ are constants. Here $x_{0}=0$ is a regular singular point.
For simplicity, we consider a second order linear ODE with a regular singular point at $x_{0}=0$. If $x_{0} \neq 0$, it is easy to convert the given ODE to an equivalent ODE with regular singular point at $x_{0}=0$. For this, we substitute $t=x-x_{0}$ and let $z(t)=y\left(x_{0}+t\right)$. Then (3) becomes

$$
t^{2} \ddot{z}+t \tilde{p}(t) \dot{z}+\tilde{q}(t) z=0
$$

where $=d / d t$. Thus, we consider following second order homogeneous linear ODE

$$
\begin{equation*}
x^{2} y^{\prime \prime}+x p(x) y^{\prime}+q(x) y=0 \tag{4}
\end{equation*}
$$

where $p, q$ are analytic at the origin.
Ordinary point vs. regular singular point: This can explained by taking two examples. Consider

$$
y^{\prime \prime}+y=0
$$

which has 0 as the ordinary point. Note that the general solution is $y=c_{1} \cos x+$ $c_{2} \sin x$. At the ordinary point $x_{0}=0$, we can find unique $c_{1}, c_{2}$ for a given $K_{0}, K_{1}$ such that $y(0)=K_{0}, y^{\prime}(0)=K_{1}$. Thus, unique solution exists for initial conditions specified at the ordinary point.
Now consider the Euler-Cauchy equation

$$
x^{2} y^{\prime \prime}-2 x y^{\prime}+2 y=0,
$$

for which $x_{0}=0$ is a regular singular point. The general solution is $y=c_{1} x+c_{2} x^{2}$. Now it is not possible to find unique values of $c_{1}, c_{2}$ for a given $K_{0}, K_{1}$ such that $y(0)=K_{0}, y^{\prime}(0)=K_{1}$. Note that solution does not exist for $K_{0} \neq 0$ since $y(0)=0$.

## 2 Frobenius method

We would like to find two linearly independent solutions of (4) so that these form a basis solution for $x \neq 0$. We find the basis solution for $x>0$. For $x<0$, we substitute $t=-x$ and carry out similar procedure for $t>0$.
If $p$ and $q$ in (4) are constants, then a solution of (4) is of the form $x^{r}$. But since $p$ and $q$ are power series, we assume that a solution of (4) can be represented by an extended power series

$$
\begin{equation*}
y=x^{r} \sum_{n=0}^{\infty} a_{n} x^{n} \tag{5}
\end{equation*}
$$

which is a product of $x^{r}$ and a power series. We also assume that $a_{0} \neq 0$. We formally substitute (5) into (4) and find $r$ and $a_{1}, a_{2}, \cdots$ in terms of $a_{0}$ and $r$. Once we find (5), we next check the convergence of the series. If it converges, then (5) becomes solution for (4).
Now from (5), we find

$$
x^{2} y^{\prime \prime}(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1) a_{n} x^{n+r}, \quad x y^{\prime}(x)=\sum_{n=0}^{\infty}(n+r) a_{n} x^{n+r} .
$$

Since $p$ and $q$ are analytic, we write

$$
p(x)=\sum_{n=0}^{\infty} p_{n} x^{n}, \quad q(x)=\sum_{n=0}^{\infty} q_{n} x^{n} .
$$

Substituting into (4), we get

$$
\sum_{n=0}^{\infty}(n+r)(n+r-1) a_{n} x^{n+r}+\left(\sum_{n=0}^{\infty} p_{n} x^{n}\right)\left(\sum_{n=0}^{\infty}(n+r) a_{n} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} q_{n} x^{n}\right)\left(\sum_{n=0}^{\infty} a_{n} x^{n+r}\right)=0 .
$$

OR

$$
x^{r} \sum_{n=0}^{\infty}\left[(r+n)(r+n-1) a_{n}+\sum_{k=0}^{n}\left((r+k) p_{n-k}+q_{n-k}\right) a_{k}\right] x^{n}=0 .
$$

Since $x>0$, this becomes

$$
\begin{equation*}
\sum_{n=0}^{\infty}\left[(r+n)(r+n-1) a_{n}+\sum_{k=0}^{n}\left((r+k) p_{n-k}+q_{n-k}\right) a_{k}\right] x^{n}=0 . \tag{6}
\end{equation*}
$$

Thus, we must have

$$
\begin{equation*}
\left[(r+n)(r+n-1)+(n+r) p_{0}+q_{0}\right] a_{n}+\sum_{k=0}^{n-1}\left[(r+k) p_{n-k}+q_{n-k}\right] a_{k}=0, \quad n=0,1,2, \cdots \tag{7}
\end{equation*}
$$

Now (7) gives $a_{n}$ in terms of $a_{0}, a_{1}, \cdots, a_{n-1}$ and $r$.
For $n=0$, we find

$$
\begin{equation*}
r(r-1)+p_{0} r+q_{0}=0, \tag{8}
\end{equation*}
$$

since $a_{0} \neq 0$. Equation (8) is called indicial equation for (4). The form of the linearly independent solutions of (4) depends on the roots of (8).
Let $\rho(r)=r(r-1)+p_{0} r+q_{0}$. Then for $n=1,2, \cdots$, we find

$$
\rho(r+n) a_{n}+b_{n}=0,
$$

where

$$
b_{n}=\sum_{k=0}^{n-1}\left[(r+k) p_{n-k}+q_{n-k}\right] a_{k} .
$$

Notice that $b_{n}$ is a linear combination of $a_{0}, a_{1}, \cdots, a_{n-1}$. Thus, we can find $a_{n}$ uniquely in terms of $r$ and $a_{0}$ if $\rho(r+n) \neq 0$. If $\rho(r+n)=0$, then it is possible to find value of $a_{n}$ in certain cases.
Let $r_{1}, r_{2}$ be the roots of the indicial equation (8). We assume that the roots are real and $r_{1} \geq r_{2}$. For $r_{1}$, clearly $\rho\left(r_{1}+n\right) \neq 0$ for $n=1,2, \cdots$. Thus, we can determine $a_{1}, a_{2}, a_{3}, \cdots$ corresponding to $r_{1}$. Clearly, one Frobenius series (extended power series) solution $y_{1}$ corresponding to the larger root $r_{1}$ always exists. Suppose $a_{0}=1$, then

$$
\begin{equation*}
y_{1}(x)=x^{r_{1}}\left(1+\sum_{n=1}^{\infty} a_{n}\left(r_{1}\right) x^{n}\right) . \tag{9}
\end{equation*}
$$

Now for $r=r_{2}$, three cases may appear. These are as follows:
A. $r_{1}-r_{2}$ is not a nonnegative integer: Then $r_{2}+n \neq r_{1}$ for any integer $n \geq 1$ and as a result $\rho\left(r_{2}+n\right) \neq 0$ for any $n \geq 1$. Thus, we can determine $a_{1}, a_{2}, a_{3}, \cdots$ corresponding to $r_{2}$. Clearly, another Frobenius series solution $y_{2}$ corresponding to the smaller root exists. Suppose $a_{0}=1$, then

$$
\begin{equation*}
y_{2}(x)=x^{r_{2}}\left(1+\sum_{n=1}^{\infty} a_{n}\left(r_{2}\right) x^{n}\right) . \tag{10}
\end{equation*}
$$

B. $r_{1}=r_{2}$, double root: Clearly a second extended power series (Frobenius series) solution does not exist.
C. $r_{1}-r_{2}=m, m \geq 1$ is a positive integer: In this case $\rho\left(r_{2}+m\right)=\rho\left(r_{1}\right)=0$.

Thus, we can find $a_{1}, a_{2}, \cdots, a_{m-1}$. But for $a_{m}$, we have

$$
\rho\left(r_{2}+m\right) a_{m}=-b_{m} .
$$

Since $\rho(r)=\left(r-r_{1}\right)\left(r-r_{2}\right)$, we have

$$
\rho(r+m)=\left(r+m-r_{1}\right)\left(r+m-r_{2}\right)=\left(r-r_{2}\right)\left(r+m-r_{2}\right) .
$$

Clearly two cases may arise here:
C.i $b_{m}$ has a factor $r-r_{2}$, i.e. $b_{m}\left(r_{2}\right)=0$. In this case, we cancel factor $r-r_{2}$ from both sides and find $a_{m}\left(r_{2}\right)$ as a finite number. Then we can continue calculating remaining coefficients $a_{m+1}, a_{m+2}, \cdots$. Hence, a second Frobenius series solution exists.
C.ii On the other hand, if $b_{m}\left(r_{2}\right) \neq 0$, then it is not possible to continue the calculations of $a_{n}$ for $n \geq m$. Hence, a second Frobenius series solution does not exist.

To find the form of the solution in the case of B and C described above, we use the reduction of order technique. We know that $y_{1}(x)$ (corresponding the larger root) always exists. Let $y_{2}(x)=v(x) y_{1}(x)$. Then

$$
\begin{aligned}
v^{\prime} & =\frac{1}{y_{1}^{2}} e^{-\int p(x) / x d x} \\
& =\frac{1}{x^{2 r_{1}}\left(1+a_{1}\left(r_{1}\right) x+a_{2}\left(r_{1}\right) x^{2}+\cdots\right)^{2}} e^{-p_{0} \ln x-p_{1} x-\cdots} \\
& =\frac{1}{x^{2 r_{1}+p_{0}}\left(1+a_{1}\left(r_{1}\right) x+a_{2}\left(r_{1}\right) x^{2}+\cdots\right)^{2}} e^{-p_{1} x-\cdots} \\
& =\frac{1}{x^{2 r_{1}+p_{0}} g(x)}
\end{aligned}
$$

where $g(x)$ is analytic at $x=0$ and $g(0)=1$. Since $g(x)$ is analytic at $x=0$ with $g(0)=1$, we must have $g(x)=1+\sum_{n=1}^{\infty} g_{n} x^{n}$. Since $r_{1}, r_{2}$ are roots of (8), we must have

$$
r_{1}+r_{2}=1-p_{0} \Rightarrow 2 r_{1}+p_{0}=m+1 .
$$

Hence,

$$
v^{\prime}=\frac{1}{x^{m+1}}+\frac{g_{1}}{x^{m}}+\cdots+\frac{g_{m-1}}{x^{2}}+\frac{g_{m}}{x}+g_{m+1}+\cdots,
$$

OR

$$
\begin{equation*}
v(x)=\frac{x^{-m}}{-m}+\frac{g_{1} x^{-m+1}}{-m+1}+\cdots+\frac{g_{m-1} x^{-1}}{-1}+g_{m} \ln x+g_{m+1} x+\cdots . \tag{11}
\end{equation*}
$$

Thus,

$$
\begin{aligned}
y_{2}(x) & =y_{1}(x)\left[\frac{x^{-m}}{-m}+\frac{g_{1} x^{-m+1}}{-m+1}+\cdots+\frac{g_{m-1} x^{-1}}{-1}+g_{m} \ln x+g_{m+1} x+\cdots .\right] \\
& =g_{m} y_{1}(x) \ln x+x^{r_{1}}\left(1+\sum_{n=1}^{\infty} a_{n}\left(r_{1}\right) x^{n}\right)\left[\frac{x^{-m}}{-m}+\frac{g_{1} x^{-m+1}}{-m+1}+\cdots+\frac{g_{m-1} x^{-1}}{-1}+g_{m+1} x+\cdots .\right]
\end{aligned}
$$

Now we take the factor $x^{-m}$ from the series inside the third bracket. Since $r_{1}-m=r_{2}$, we finally find

$$
\begin{equation*}
y_{2}(x)=c y_{1}(x) \ln x+x^{r_{2}} \sum_{n=0}^{\infty} c_{n} x^{n} \tag{12}
\end{equation*}
$$

where we put $g_{m}=c$.
Now for $r_{1}=r_{2}$, we have $m=0$ and hence $g_{m}=g_{0}=g(0)=1=c$. Thus, $\ln x$ term is definitely present in the second solution. Also in this case, the series in (11) starts with $g_{0} \ln x$ and the next term is $g_{1} x$. Hence, for $r_{1}=r_{2}$, we must have $c_{0}=0$ in (12). In certain cases, $g_{m}=c$ becomes zero (case C.ii) for $m \geq 1$. Then the second solution is also a Frobenius series solution; otherwise, the second Frobenius series solution does not exist.

## 3 Summary

The results derived in the previous section can be summarized as follows. Consider

$$
\begin{equation*}
x^{2} y^{\prime \prime}+x p(x) y^{\prime}+q(x) y=0 \tag{13}
\end{equation*}
$$

where $p$ and $q$ have convergent power series expansion in $|x|<R, R>0$. Let $r_{1}, r_{2}$ ( $r_{1} \geq r_{2}$ ) be the roots of the indicial equation:

$$
\begin{equation*}
r^{2}+(p(0)-1) r+q(0)=0 \tag{14}
\end{equation*}
$$

For $x>0$ we have the following theorems:
Theorem 1. If $r_{1}-r_{2}$ is not zero or a positive integer, then there are two linearly independent solutions $y_{1}$ and $y_{2}$ of (13) of the form

$$
\begin{equation*}
y_{1}(x)=x^{r_{1}} \sigma_{1}(x), \quad y_{2}(x)=x^{r_{2}} \sigma_{2}(x), \tag{15}
\end{equation*}
$$

where $\sigma_{1}, \sigma_{2}$ are analytic at $x=0$ with radius of convergence $R$ and $\sigma_{1}(0) \neq 0$ and $\sigma_{2}(0) \neq 0$.

Theorem 2. If $r_{1}=r_{2}$, then there are two linearly independent solutions $y_{1}$ and $y_{2}$ of (13) of the form

$$
\begin{equation*}
y_{1}(x)=x^{r_{1}} \sigma_{1}(x), \quad y_{2}(x)=(\ln x) y_{1}(x)+x^{r_{2}+1} \sigma_{2}(x), \tag{16}
\end{equation*}
$$

where $\sigma_{1}, \sigma_{2}$ are analytic at $x=0$ with radius of convergence $R$ and $\sigma_{1}(0) \neq 0$.
Theorem 3. If $r_{1}-r_{2}$ is a positive integer, then there are two linearly independent solutions $y_{1}$ and $y_{2}$ of (13) of the form

$$
\begin{equation*}
y_{1}(x)=x^{r_{1}} \sigma_{1}(x), \quad y_{2}(x)=c(\ln x) y_{1}(x)+x^{r_{2}} \sigma_{2}(x), \tag{17}
\end{equation*}
$$

where $\sigma_{1}, \sigma_{2}$ are analytic at $x=0$ with radius of convergence $R$ and $\sigma_{1}(0) \neq 0$ and $\sigma_{2}(0) \neq 0$. It may happen that $c=0$.

Example 4. Discuss whether two Frobenius series solutions exist or do not exist for the following equations:
(i) $2 x^{2} y^{\prime \prime}+x(x+1) y^{\prime}-(\cos x) y=0$,
(ii) $x^{4} y^{\prime \prime}-\left(x^{2} \sin x\right) y^{\prime}+2(1-\cos x) y=0$.

Solution: (i) We can write this as

$$
x^{2} y^{\prime \prime}+\frac{(x+1)}{2} x y^{\prime}-\frac{\cos x}{2} y=0 .
$$

Hence $p(x)=(x+1) / 2$ and $q(x)=-\cos x / 2$. Thus, $p(0)=1 / 2$ and $q(0)=-1 / 2$. The indicial equation is

$$
r^{2}+(p(0)-1) r+q(0)=0 \Rightarrow 2 r^{2}-r-1=0 \Rightarrow r_{1}=1, r_{2}=-1 / 2
$$

Since $r_{1}-r_{2}=3 / 2$, which is not zero or a positive integer, two Frobenius series solutions exist.
(ii) We can write this as

$$
x^{2} y^{\prime \prime}-\frac{\sin x}{x} x y^{\prime}+2 \frac{1-\cos x}{x^{2}} y=0 .
$$

Hence $p(x)=-\sin x / x$ and $q(x)=2(1-\cos x) / x^{2}$. Thus, $p(0)=-1$ and $q(0)=1$. The indicial equation is

$$
r^{2}+(p(0)-1) r+q(0)=0 \Rightarrow r^{2}-2 r+1=0 \Rightarrow r_{1}=1=r_{2} .
$$

Since $r_{1}=r_{2}$, only one Frobenius series solutions exists.
Example 5. (Case A) Find two independent solutions around $x=0$ for

$$
2 x y^{\prime \prime}+(x+1) y^{\prime}+3 y=0
$$

Solution: We write this as

$$
x^{2} y^{\prime \prime}+\frac{(x+1)}{2} x y^{\prime}+(3 x / 2) y=0 .
$$

Hence $p(x)=(x+1) / 2$ and $q(x)=3 x / 2$. Thus, $p(0)=1 / 2, q(0)=0$. The indicial equation is

$$
r^{2}+(p(0)-1) r+q(0)=0 \Rightarrow 2 r^{2}-r=0 \Rightarrow r_{1}=1 / 2, r_{2}=0
$$

Since $r_{1}-r_{2}=1 / 2$, is not zero or a positive integer, two independent Frobenius series solution exist.
Substituting

$$
y=x^{r} \sum_{n=0}^{\infty} a_{n} x^{n}
$$

(after some manipulation and cancelling $x^{r}$ ) we find

$$
\sum_{n=0}^{\infty} \rho(n+r) a_{n} x^{n}+\sum_{n=1}^{\infty}((n+r-1)+3) a_{n-1} x^{n}=0
$$

where $\rho(r)=r(2 r-1)$. Rearranging the above, we get

$$
\rho(r) a_{0}+\sum_{n=1}^{\infty}\left[\rho(n+r) a_{n}+(n+r+2) a_{n-1}\right] x^{n}=0
$$

Hence, we find (since $a_{0} \neq 0$ )

$$
\rho(r)=0, \quad \rho(n+r) a_{n}+(n+r+2) a_{n-1}=0 \text { for } n \geq 1 .
$$

From the first relation we find roots of the indicial equation $r_{1}=1 / 2, r_{2}=0$. Now with the larger root $r=r_{1}=1 / 2$, we find

$$
a_{n}=-\frac{(2 n+5) a_{n-1}}{2 n(2 n+1)}, \quad n \geq 1
$$

Iterating we find

$$
a_{1}=-\frac{7}{6} a_{0}, \quad a_{2}=\frac{21}{40} a_{0}, \cdots
$$

Hence, by induction

$$
a_{n}=(-1)^{n} \frac{(2 n+5)(2 n+3)}{15 \cdot 2^{n} n!} a_{0}, \quad n \geq 1 \quad \text { (Check!) }
$$

Thus, taking $a_{0}=1$, we find

$$
y_{1}(x)=x^{1 / 2}\left(1-\frac{7}{6} x+\frac{21}{40} x^{2}-\cdots\right)
$$

Now with $r=r_{2}=0$, we find

$$
a_{n}=-\frac{(n+2) a_{n-1}}{n(2 n-1)}, \quad n \geq 1
$$

Iterating we find

$$
a_{1}=-3 a_{0}, \quad a_{2}=2 a_{0}, \cdots
$$

Hence, by induction

$$
\begin{equation*}
a_{n}=(-1)^{n}\left(\frac{5}{2 n-1}-\frac{2}{n}\right)\left(\frac{5}{2 n-3}-\frac{2}{n-1}\right) \cdots\left(\frac{5}{1}-\frac{2}{1}\right) a_{0}, \quad n \geq 1 \tag{Check!}
\end{equation*}
$$

Thus, taking $a_{0}=1$, we find

$$
y_{2}(x)=\left(1-3 x+2 x^{2}-\cdots\right)
$$

Example 6. (Case B) Find the general solution in the neighbourhood of origin for

$$
4 x^{2} y^{\prime \prime}-8 x^{2} y^{\prime}+\left(4 x^{2}+1\right) y=0
$$

Solution: We write this as

$$
x^{2} y^{\prime \prime}-(2 x) x y^{\prime}+\left(x^{2}+1 / 4\right) y=0
$$

Hence $p(x)=-2 x$ and $q(x)=x^{2}+1 / 4$. Thus, $p(0)=0, q(0)=1 / 4$. The indicial equation is

$$
r^{2}+(p(0)-1) r+q(0)=0 \Rightarrow r^{2}-r+1 / 4=0 \Rightarrow r_{1}=r_{2}=1 / 2
$$

Since the indicial equation has a double root, only one Frobenius series solution exists. Substituting

$$
y=x^{r} \sum_{n=0}^{\infty} a_{n} x^{n}
$$

(after some manipulation and cancelling $x^{r}$ ) we find

$$
\sum_{n=0}^{\infty} \rho(n+r) a_{n} x^{n}-\sum_{n=1}^{\infty} 8(n+r-1) a_{n-1} x^{n}+\sum_{n=2}^{\infty} 4 a_{n-2} x^{n}=0
$$

where $\rho(r)=(2 r-1)^{2}$. Rearranging the above, we get

$$
\rho(r) a_{0}+\left(\rho(r+1) a_{1}-8 r a_{0}\right) x+\sum_{n=2}^{\infty}\left[\rho(n+r) a_{n}-8(n+r-1) a_{n-1}+4 a_{n-2}\right] x^{n}=0 .
$$

Now with $r=1 / 2$, we find

$$
a_{1}=a_{0}, a_{n}=\frac{(2 n-1) a_{n-1}}{n^{2}}-\frac{a_{n-2}}{n^{2}}, \quad n \geq 2
$$

Iterating we find

$$
a_{2}=\frac{1}{2!} a_{0}, \quad a_{3}=\frac{1}{3!} a_{0}, \quad a_{4}=\frac{1}{4!} a_{0}, \cdots
$$

Hence, by induction

$$
a_{n}=\frac{1}{n!} a_{0}, \quad n \geq 1
$$

$\left\{\right.$ Induction: Claim $a_{k}=a_{0} / k!$. True for $k=1,2$. Assume it is true for $k=m$. Now for $k=m+1$,

$$
\left.a_{k+1}=\frac{(2 k+1) a_{k}}{(k+1)^{2}} a_{0}-\frac{a_{k-1}}{(k+1)^{2}} a_{0}=\frac{1}{(k-1)!(k+1)^{2}} \frac{k+1}{k} a_{0}=\frac{a_{0}}{(k+1)!}\right\}
$$

Thus, taking $a_{0}=1$, we find

$$
y_{1}(x)=x^{1 / 2}\left(1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots\right)=x^{1 / 2} e^{x}
$$

For the general solution, we need to find another solution $y_{2}$. For this we use reduction of order. Let $y_{2}(x)=y_{1}(x) v(x)$. Then

$$
v=\int \frac{1}{y_{1}^{2}} e^{-\int p d x} d x
$$

where $p(x)=-2$. Hence

$$
v(x)=\int \frac{1}{x} d x=\ln x
$$

and $y_{2}=(\ln x) x^{1 / 2} e^{x}$. Thus, the general solution is

$$
y(x)=x^{1 / 2} e^{x}\left(c_{1}+c_{2} \ln x\right)
$$

Example 7. (Case C.i) Find two independent solutions around $x=0$ for

$$
x y^{\prime \prime}+2 y^{\prime}+x y=0
$$

Solution: We write this as

$$
x^{2} y^{\prime \prime}+2 x y^{\prime}+x^{2} y=0
$$

Hence $p(x)=2$ and $q(x)=x^{2}$. Thus, $p(0)=2, q(0)=0$. The indicial equation is

$$
r^{2}+(p(0)-1) r+q(0)=0 \Rightarrow r^{2}+r=0 \Rightarrow r_{1}=0, r_{2}=-1 .
$$

A Frobenius series solution exists for the larger root $r_{1}=0$. Substituting

$$
y=x^{r} \sum_{n=0}^{\infty} a_{n} x^{n}
$$

(after some manipulation and cancelling $x^{r}$ ) we find

$$
\sum_{n=0}^{\infty} \rho(n+r) a_{n} x^{n}+\sum_{n=2}^{\infty} a_{n-2} x^{n}=0
$$

where $\rho(r)=r(r+1)$. Rearranging the above, we get

$$
\rho(r) a_{0}+\rho(r+1) a_{1} x+\sum_{n=2}^{\infty}\left[\rho(n+r) a_{n}+a_{n-2}\right] x^{n}=0 .
$$

Hence, we find (since $a_{0} \neq 0$ )

$$
\rho(r)=0, \quad \rho(r+1) a_{1}=0, \quad \rho(n+r) a_{n}+a_{n-2}=0 \quad \text { for } n \geq 2 .
$$

From the first relation we find roots of the indicial equation $r_{1}=0, r_{2}=-1$. Now with the larger root $r=r_{1}$, we find

$$
a_{1}=0, a_{n}=-\frac{a_{n-2}}{n(n+1)}, \quad n \geq 2 .
$$

Iterating we find

$$
a_{2}=-\frac{1}{3!} a_{0}, \quad a_{3}=0, \quad a_{4}=\frac{1}{5!} a_{0}, \cdots
$$

Hence, by induction

$$
a_{2 n}=(-1)^{n} \frac{1}{(2 n+1)!} a_{0}, \quad a_{2 n+1}=0 .
$$

Thus, taking $a_{0}=1$, we find

$$
y_{1}(x)=\left(1-\frac{x^{2}}{3!}+\frac{x^{4}}{5!}-\cdots\right)=\frac{\sin x}{x}
$$

Since $r_{1}-r_{2}=1$, a positive integer, the second Frobenius series solution may or may not exist. Hence, to be sure, we need to compute it. With $r=r_{2}=-1$, we find

$$
0 \cdot a_{1}=0, \quad a_{n}=-\frac{a_{n-2}}{n(n-1)}, \quad n \geq 2
$$

Now the first relation can be satisfied by taking any value of $a_{1}$. For simplicity, we choose $a_{1}=0$. Iterating we find

$$
a_{2}=-\frac{1}{2!} a_{0}, \quad a_{3}=0, \quad a_{4}=\frac{1}{4!} a_{0}, \cdots
$$

Hence, by induction

$$
a_{2 n}=(-1)^{n} \frac{1}{(2 n)!} a_{0}, \quad a_{2 n+1}=0
$$

Thus, indeed a second Frobenius series solution exists and taking $a_{0}=1$, we get

$$
y_{2}(x)=x^{-1}\left(1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\cdots\right)=\frac{\cos x}{x} .
$$

Comment: The second solution could have been obtained using reduction of order also. Suppose $y_{2}=v y_{1}$, then

$$
v=\int \frac{x^{2}}{\sin ^{2} x} e^{-\int 2 / x d x} d x=\int \operatorname{cosec}^{2} x d x=-\cot x
$$

Hence $y_{2}(x)=\cos x / x$ (disregarding minus sign)
Example 8. (Case C.ii) Find general solution around $x=0$ for

$$
\left(x^{2}-x\right) y^{\prime \prime}-x y^{\prime}+y=0
$$

Solution: We write this as

$$
x^{2} y^{\prime \prime}-\frac{x}{x-1} x y^{\prime}+\frac{x}{x-1} y=0 .
$$

Hence $p(x)=-x /(x-1)$ and $q(x)=x /(x-1)$. Thus, $p(0)=q(0)=0$. The indicial equation is

$$
r^{2}+(p(0)-1) r+q(0)=0 \Rightarrow r^{2}-r=0 \Rightarrow r_{1}=1, r_{2}=0
$$

Since $r_{1}-r_{2}=1$, a positive integer, two independent Frobenius series solution may or may not exist.
Substituting

$$
y=x^{r} \sum_{n=0}^{\infty} a_{n} x^{n}
$$

(after some manipulation and cancelling $x^{r}$ ) we find

$$
(x-1) \sum_{n=0}^{\infty}(n+r)(n+r-1) a_{n} x^{n}-x \sum_{n=0}^{\infty}(n+r) a_{n} x^{n}+x \sum_{n=0}^{\infty} a_{n} x^{n}=0
$$

Rearranging the above, we get

$$
(x-1) \sum_{n=0}^{\infty}(n+r)(n+r-1) a_{n} x^{n}-x \sum_{n=0}^{\infty}((n+r)-1) a_{n} x^{n}=0
$$

OR

$$
x \sum_{n=0}^{\infty}(n+r-1)^{2} a_{n} x^{n}-\sum_{n=0}^{\infty}(n+r)(n+r-1) a_{n} x^{n}=0 .
$$

OR

$$
\sum_{n=1}^{\infty}(n+r-2)^{2} a_{n-1} x^{n}-\sum_{n=0}^{\infty}(n+r)(n+r-1) a_{n} x^{n}=0
$$

OR

$$
r(r-1) a_{0}+\sum_{n=1}^{\infty}\left[(n+r)(n+r-1) a_{n}-(n+r-2)^{2} a_{n-1}\right] x^{n}=0
$$

Hence, we find (since $a_{0} \neq 0$ )

$$
\rho(r)=0, \quad \rho(n+r) a_{n}-(n+r-2)^{2} a_{n-1}=0 \text { for } n \geq 1,
$$

where $\rho(r)=r(r-1)$. From the first relation we find roots of the indicial equation $r_{1}=1, r_{2}=0$. Now with the larger root $r=r_{1}=1$, we find

$$
a_{n}=\frac{(n-1)^{2} a_{n-1}}{n(n+1)}, \quad n \geq 1
$$

Iterating we find

$$
a_{n}=0, \quad n \geq 1 .
$$

Thus, taking $a_{0}=1$, we find

$$
y_{1}(x)=x
$$

Now with $r=r_{2}=0$, we find

$$
n(n-1) a_{n}=(n-2)^{2} a_{n-1}, \quad n \geq 1 .
$$

Now for $n=1$, we find $0=a_{0}$ which is a contradiction. Hence, second Frobenius series solution does not exist. To find the second independent solution, we use reduction of order technique. Let $y_{2}(x)=v(x) y_{1}(x)$. Then

$$
v(x)=\int \frac{1}{y_{1}^{2}} e^{-\int p d x} d x
$$

where $p(x)=-x /\left(x^{2}-x\right)=-1 /(x-1)$. Hence,

$$
v(x)=\int \frac{1}{x^{2}} e^{\ln (1-x)} d x=\int\left(\frac{1}{x^{2}}-\frac{1}{x}\right) d x=-\left(\frac{1}{x}+\ln x\right) .
$$

(Why I wrote $\ln (1-x)$ NOT $\ln (x-1)$ ?) Hence, $y_{2}(x)=(1+x \ln x)$ (disregarding the minus sign, since the ODE is homogeneous and linear). Thus, the general solution is given by

$$
y(x)=c_{1} x+c_{2}(1+x \ln x) .
$$

