

1 Gamma function

*Gamma function* is defined by

\[ \Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt, \quad p > 0. \] (1)

The integral in (1) is convergent that can be proved easily. Some special properties of the gamma function are the following:

i. It is readily seen that \( \Gamma(p + 1) = p\Gamma(p) \), since

\[
\Gamma(p + 1) = \lim_{T \to \infty} \int_0^T e^{-t} t^p dt \\
= \lim_{T \to \infty} \left[ -e^{-t} t^p \bigg|_0^T + p \int_0^T e^{-t} t^{p-1} dt \right] \\
= p \int_0^\infty e^{-t} t^{p-1} dt = p\Gamma(p).
\]

ii. \( \Gamma(1) = 1 \) (trivial proof)

iii. If \( p = m \), a positive integer, then \( \Gamma(m + 1) = m! \) (use i. repeatedly)

iv. \( \Gamma(1/2) = \sqrt{\pi} \). This can be proved as follows:

\[ I = \Gamma(1/2) = \int_0^\infty e^{-t^{-1/2}} dt = 2 \int_0^\infty e^{-u^2} du. \]

Hence \[ I^2 = 4 \int_0^\infty e^{-u^2} du \int_0^\infty e^{-v^2} dv = 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy. \]

Using polar coordinates \( \rho, \theta \), the above becomes

\[ I^2 = 4 \int_0^\infty \int_0^{\pi/2} e^{-\rho^2} \rho d\rho d\theta \Rightarrow I^2 = \pi \Rightarrow I = \sqrt{\pi} \]

v. Using relation in i., we can extend the definition of \( \Gamma(p) \) for \( p < 0 \). Suppose \( N \) is a positive integer and \( -N < p < -N + 1 \). Now using relation of i., we find

\[ \Gamma(p) = \frac{\Gamma(p + 1)}{p} = \frac{\Gamma(p + 2)}{p(p + 1)} = \cdots = \frac{\Gamma(p + N)}{p(p + 1) \cdots (p + N - 1)}. \]

Since \( p + N > 0 \), the above relation is well defined.

vi. \( \Gamma(p) \) is not defined when \( p \) is zero or a negative integer. For small positive \( \epsilon \),

\[ \Gamma(\pm \epsilon) = \frac{\Gamma(1 \pm \epsilon)}{\pm \epsilon} \approx \frac{1}{\pm \epsilon} \to \pm \infty \quad \text{as} \quad \epsilon \to 0. \]

Since \( \Gamma(0) \) is undefined, \( \Gamma(p) \) is also undefined when \( p \) is a negative integer.
Bessel’s equation

Bessel’s equation of order $\nu$ ($\nu \geq 0$) is given by

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \quad (2)$$

Obviously, $x = 0$ is regular singular point. Since $p(0) = 1, q(0) = -\nu^2$, the indicial equation is given by

$$r^2 - \nu^2 = 0.$$

Hence, $r_1 = \nu, r_2 = -\nu$ and $r_1 - r_2 = 2\nu$. A Frobenius series solution exists for the larger root $r = r_1 = \nu$. To find this series, we substitute

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \quad x > 0$$

into (2) and (after some manipulation) find

$$\sum_{n=0}^{\infty} \rho(n + r)a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

where $\rho(r) = r^2 - \nu^2$. This equation is rearranged as

$$\rho(r)a_0 + \rho(r + 1)a_1 x + \sum_{n=2}^{\infty} (\rho(n + r)a_n + a_{n-2})x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0, \quad \rho(r + 1)a_1 = 0, \quad \rho(r + n)a_n = -a_{n-2}, \quad n \geq 2.$$

From the first relation, we get $r_1 = \nu, r_2 = -\nu$. Now with the larger root $r = r_1$ we find

$$a_1 = 0, \quad a_n = -\frac{a_{n-2}}{n(n + 2\nu)}, \quad n \geq 2.$$

Iterating we find (by induction),

$$a_{2n+1} = 0, \quad a_{2n} = (-1)^n \frac{1}{2^{2n+1} n!(\nu + 1)(\nu + 2) \cdots (\nu + n)} a_0, \quad n \geq 1.$$

Hence

$$y_1(x) = a_0 x^\nu \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+1} n!(\nu + 1)(\nu + 2) \cdots (\nu + n)} \right). \quad (3)$$

Here it is usual to choose (instead of $a_0 = 1$ as was done in lecture 14)

$$a_0 = \frac{1}{2^\nu \Gamma(\nu + 1)}.$$

Then the Frobenius series solution (3) is called the Bessel function of order $\nu$ of the first kind and is denoted by $J_\nu(x)$:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+\nu}. \quad (4)$$

To find the second independent solution, we consider the following three cases:
A. \( r_1 - r_2 = 2\nu \) is not a nonnegative integer: We know that a second Frobenius series solution for \( r_2 = -\nu \) exist. We do similar calculation as in the case of \( r_1 \) and it turns out that the resulting series is given by (4) with \( \nu \) replaced by \(-\nu\). Hence, the second solution is given by
\[
J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n - \nu + 1)} \left( \frac{x}{2} \right)^{2n-\nu}.
\] (5)

B. \( r_1 = r_2 \): Obviously this corresponds to \( \nu = 0 \) and a second Frobenius series solution does not exist.

C. \( r_1 - r_2 = 2\nu \) is a positive integer: Now there are two cases. We discuss them separately.

C.i \( \nu \) is not a positive integer: Clearly \( \nu = (2k+1)/2 \), where \( k \in \{0, 1, 2, \cdots\} \). Now we have found earlier that (since \( a_0 \neq 0 \))
\[
\rho(r) = 0, \quad \rho(r+1)a_1 = 0, \quad \rho(r+n)a_n = -a_{n-2}, \quad n \geq 2.
\]
With \( r = r_2 = -\nu \), we get
\[
\rho(r) = 0; \quad 1 \cdot (1 - (2k+1))a_1 = 0; \quad n \cdot (n - (2k+1))a_n = -a_{n-2}, \quad n \geq 2.
\]
It is clear that the even terms \( a_{2n} \) can be determined uniquely. For odd terms, \( a_1 = a_3 = \cdots = a_{2k-1} = 0 \) but for \( a_{2k+1} \) we must have
\[
\begin{align*}
n \cdot 0 \cdot a_{2k+1} &= -a_{2k-1} \Rightarrow 0 \cdot a_{2k+1} = 0.
\end{align*}
\]
This can be satisfied by taking any value of \( a_{2k+1} \) and for simplicity, we can take \( a_{2k+1} = 0 \). Rest of the odd terms thus also vanish. Hence, the second solution in this case is also given by (5), i.e.
\[
J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n - \nu + 1)} \left( \frac{x}{2} \right)^{2n-\nu}.
\] (6)

C.ii \( \nu \) is a positive integer: Clearly \( \nu = k \), where \( k \in \{1, 2, 3, \cdots\} \). Now we find (since \( a_0 \neq 0 \))
\[
\rho(r) = 0, \quad \rho(r+1)a_1 = 0, \quad \rho(r+n)a_n = -a_{n-2}, \quad n \geq 2.
\]
With \( r = r_2 = -\nu \), we get
\[
\rho(r) = 0; \quad 1 \cdot (1 - 2k)a_1 = 0; \quad n \cdot (n - 2k)a_n = -a_{n-2}, \quad n \geq 2.
\]
It is clear that all the odd terms \( a_{2n+1} \) vanish. For even terms, \( a_2, a_4, \cdots, a_{2k-2} \) each is nonzero. For \( a_{2k} \) we must have
\[
\begin{align*}
n \cdot 0 \cdot a_{2k} &= -a_{2k-2} \Rightarrow 0 \cdot a_{2k} \neq 0,
\end{align*}
\]
which is a contradiction. Thus, a second Frobenius series solution does not exist in this case.
Summary of solutions for Bessel’s equation: The Bessel’s equation of order \( \nu \) \((\nu \geq 0)\)
\[
x^2 y'' + xy' + (x^2 - \nu^2)y = 0,
\]
has two independent Frobenius series solutions \( J_\nu \) and \( J_{-\nu} \) when \( \nu \) is not an (nonnegative) integer:
\[
J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+\nu}, \quad J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n - \nu + 1)} \left(\frac{x}{2}\right)^{2n-\nu}.
\]
Thus the general solution, when \( \nu \) is not an (nonnegative) integer, is
\[
y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x).
\]
When \( \nu \) is a (nonnegative) integer, a second solution, which is independent of \( J_\nu \), can be found. This solution is called Bessel function of second kind and is denoted by \( Y_\nu \). Hence, the general solution, when \( \nu \) is an (nonnegative) integer, is
\[
y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x).
\]

3 Linear dependence of \( J_m \) and \( J_{-m} \), \( m \) is a +ve integer

When \( \nu = m \) is a positive integer, then
\[
J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + m + 1)} \left(\frac{x}{2}\right)^{2n+m} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{x}{2}\right)^{2n+m},
\]
since \( \Gamma(n + m + 1) = (n+m)! \).
Since \( \Gamma(\pm 0) = \pm\infty \), we define \( 1/\Gamma(k) \) to be zero when \( k \) is nonpositive integer. Now
\[
J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n - m + 1)} \left(\frac{x}{2}\right)^{-2n-m}.
\]
Now each term in the sum corresponding to \( n = 0 \) to \( n = m - 1 \) is zero since \( 1/\Gamma(k) \) is zero when \( k \) is nonpositive integer. Hence, we write the sum starting from \( n = m \):
\[
J_{-m}(x) = \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\Gamma(n - m + 1)} \left(\frac{x}{2}\right)^{-2n+m}.
\]
Substituting \( n - m = k \), we find
\[
J_{-m}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+m}}{(m+k)!\Gamma(k+1)} \left(\frac{x}{2}\right)^{2(m+k)-m}
= (-1)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m}
= (-1)^m J_m(x).
\]
Hence \( J_m \) and \( J_{-m} \) becomes linearly dependent when \( m \) is a positive integer.
4 Properties of Bessel function

Few important relationships are very useful in application. These are described here.

A. From the expression for $J_\nu$ given in (4), we find

$$x^\nu J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+2\nu}$$

Taking derivative with respect to $x$ we find

$$\left(x^\nu J_\nu(x)\right)' = \sum_{n=0}^{\infty} \frac{(-1)^n (n + \nu)}{n!\Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+2\nu-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \nu)} \left(\frac{x}{2}\right)^{2n+2\nu-1},$$

where we have used $\Gamma(n + \nu + 1) = (n + \nu)\Gamma(n + \nu)$. We can write the above relation as

$$\left(x^\nu J_\nu(x)\right)' = x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + (\nu - 1) + 1)} \left(\frac{x}{2}\right)^{2n+\nu-1}.$$

Hence,

$$\left(x^\nu J_\nu(x)\right)' = x^\nu J_{\nu-1}(x). \quad (7)$$

B. From (4), we find

$$x^{-\nu} J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu}n!\Gamma(n + \nu + 1)} x^{2n}.$$

Taking derivative with respect to $x$ we find

$$\left(x^{-\nu} J_\nu(x)\right)' = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+\nu-1}(n - 1)!\Gamma(n + \nu + 1)} x^{2n-1}.$$

Note that the sum runs from $n = 1$ (in contrast to that in A). Let $k = n - 1$, then we obtain

$$\left(x^{-\nu} J_\nu(x)\right)' = x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!\Gamma(k + (\nu + 1) + 1)} \left(\frac{x}{2}\right)^{2k+\nu+1}$$

$$= -x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k + (\nu + 1) + 1)} \left(\frac{x}{2}\right)^{2k+\nu+1}.$$

Hence,

$$\left(x^{-\nu} J_\nu(x)\right)' = -x^{-\nu} J_{\nu+1}(x). \quad (8)$$

Note: In the first relation A, while taking derivative, we keep the sum running from $n = 0$. This is true only when $\nu > 0$. In the second relation B, we only need $\nu \geq 0$. Taking $\nu = 0$ in B, we find $J_0' = -J_1$. If we put $\nu = 0$ in A, then we find $J_0' = J_{-1}$. But $J_{-1} = -J_1$ and hence we find the same relation as that in B. Hence, the first relation is also valid for $\nu \geq 0$. 
C. From A and B, we get

\[ J'_\nu + \frac{\nu}{x} J_\nu = J_{\nu-1} \]
\[ J'_\nu - \frac{\nu}{x} J_\nu = -J_{\nu+1} \]

Adding and subtracting we find

\[ J_{\nu-1} - J_{\nu+1} = 2J'_\nu \] \hspace{1cm} (9)

and

\[ J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu. \] \hspace{1cm} (10)