1 Introduction

Differential equations, whether ordinary or partial, describe the ways certain quantities of interest vary over time. These equations are generally coupled with initial conditions at time $t = 0$ and boundary conditions.

Laplace transform is a powerful technique to solve differential equations. It transforms an IVP in ODE to algebraic equations. The solution of the algebraic equations is then back-transformed to the original problem. In case of PDE, it can be applied to any independent variable $x, y, z$ and $t$ that varies from $0$ to $\infty$. After applying Laplace transform, the original PDE becomes a new PDE with one less independent variable or an ODE. The resulting problem can be solved by other method (such as separation of variables, another transform) and the solution is again back-transformed to the original problem.

2 Laplace transform

Let a function $f$ be defined for $t \geq 0$. We define the Laplace transform of $f$, denoted by $F(s)$ or $\mathcal{L}\left(f(t)\right)$, as

$$F(s) = \mathcal{L}\left(f(t)\right) = \int_0^{\infty} e^{-st} f(t) \, dt,$$

for those $s$ for which the integral in (1) exists. We also refer $f(t)$ as the inverse Laplace transform of $F(s)$ and we write

$$f(t) = \mathcal{L}^{-1}\left(F(s)\right).$$

Comment 1: Laplace transform is defined for complex valued function $f(t)$ and the parameter $s$ can also be complex. But we restrict our discussion only for the case in which $f(t)$ is real valued and $s$ is real.

Comment 2: Since the integral in (1) is an improper integral, existence of Laplace transform implies that the following limit exists:

$$F(s) = \int_0^{\infty} e^{-st} f(t) \, dt = \lim_{A \to \infty} \int_0^{A} e^{-st} f(t) \, dt.$$

Example 1. Consider the function defined by

$$f(t) = \begin{cases} 
0, & 0 \leq t < a, \\
1, & t > a.
\end{cases}$$
Now

\[ F(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt \]
\[ = \int_{a}^{\infty} e^{-st} dt \]
\[ = \frac{e^{-as}}{s}, \quad s > 0. \]

**Example 2.** Consider \( f(t) = t^a, \, a > -1. \) Now

\[ F(s) = \int_{0}^{\infty} e^{-st} t^a \, dt \]
\[ = \frac{1}{s^{a+1}} \int_{0}^{\infty} e^{-u} u^{a+1-1} \, du \]
\[ = \frac{\Gamma(a+1)}{s^{a+1}}, \quad s > 0. \]

Hence \( F(1) = 1/s; \, F(t) = 1/s^2; \, F(t^n) = n!/s^{n+1}, \) where \( n \) is nonnegative integer.

**Example 3.** Consider \( f(t) = e^{at}. \) Now

\[ F(s) = \int_{0}^{\infty} e^{-st} e^{at} \, dt \]
\[ = \int_{0}^{\infty} e^{-(s-a)t} \, dt \]
\[ = \frac{1}{s-a}, \quad s > a. \]

**Example 4.** Consider \( f(t) = \cos(\omega t). \) Using

\[ \int \cos(at) e^{bt} \, dt = \frac{e^{bt}}{a^2 + b^2} \left( b \cos(at) + a \sin(at) \right) \]

\[ F(s) = \int_{0}^{\infty} e^{-st} \cos(\omega t) \, dt \]
\[ = \frac{s}{s^2 + \omega^2}, \quad s > 0. \]

**Example 5.** Consider \( f(t) = \sin(\omega t). \) Using

\[ \int \sin(at) e^{bt} \, dt = \frac{e^{bt}}{a^2 + b^2} \left( b \sin(at) - a \cos(at) \right) \]

\[ F(s) = \int_{0}^{\infty} e^{-st} \sin(\omega t) \, dt \]
\[ = \frac{\omega}{s^2 + \omega^2}, \quad s > 0. \]
2.1 Existence of Laplace transform

We give sufficient condition for the existence of LT. We need the concept of piecewise continuous function.

**Definition 1.** *(Piecewise continuous function)* A function $f$ is piecewise continuous on the interval $[a,b]$ if

(i) The interval $[a,b]$ can be broken into a finite number of subintervals $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, such that $f$ is continuous in each subinterval $(t_i, t_{i+1})$, for $i = 0, 1, 2, \cdots, n-1$

(ii) The function $f$ has jump discontinuity at $t_i$, thus

\[
\lim_{t \to t_i^+} f(t) < \infty, \quad i = 0, 1, 2, \cdots, n-1; \quad \lim_{t \to t_i^-} f(t) < \infty, \quad i = 1, 2, 3, \cdots, n.
\]

**Note:** A function is piecewise continuous on $[0, \infty)$ if it is piecewise continuous in $[0, A]$ for all $A > 0$

**Example 6.** The function defined by

\[
f(t) = \begin{cases} 
  t^2, & 0 \leq t \leq 1, \\
  3 - t, & 1 < t \leq 2, \\
  t + 1, & 2 < t \leq 3,
\end{cases}
\]

is piecewise continuous on $[0, 3]$

**Example 7.** The function defined by

\[
f(t) = \begin{cases} 
  \frac{1}{2 - t}, & 0 \leq t < 2, \\
  t + 1, & 2 \leq t \leq 3,
\end{cases}
\]

is NOT piecewise continuous on $[0, 3]$

**Definition 2.** *(Exponential order)* A function $f$ is said to be of exponential order if there exist constants $M$ and $c$ such that

\[|f(t)| \leq Me^{ct} \quad \text{for sufficiently large } t.\]

**Example 8.** Any polynomial is of exponential order. This is clear from the fact that

\[
e^{at} = \sum_{n=0}^{\infty} \frac{t^n a^n}{n!} \geq \frac{t^n a^n}{n!} \quad \Rightarrow \quad t^n \leq \frac{n!}{a^n} e^{at}
\]

But $f(t) = e^{t^2}$ is not of exponential order.

**Sufficient condition for the existence of Laplace transform:** Let $f$ be a piecewise continuous function in $[0, \infty)$ and is of exponential order. Then Laplace transform $F(s)$ of $f$ exists for $s > c$, where $c$ is a real number that depends on $f$. 
Proof: Since $f$ is of exponential order, there exists $A, M, c$ such that
$$|f(t)| \leq Me^{ct} \quad \text{for } t \geq A.$$ 

Now we write
$$I = \int_0^\infty f(t)e^{-st} \, dt = I_1 + I_2,$$
where
$$I_1 = \int_0^A f(t)e^{-st} \, dt \quad \text{and} \quad I_2 = \int_A^\infty f(t)e^{-st} \, dt.$$

Since $f$ is piecewise continuous, $I_1$ exists. For the second integral $I_2$, we note that for $t \geq A$
$$|e^{-st}f(t)| \leq Me^{-(s-c)t}.$$

Thus
$$\int_A^\infty |f(t)e^{-st}| \, dt \leq M \int_A^\infty e^{-(s-c)t} \, dt \leq M \int_0^\infty e^{-(s-c)t} \, dt = \frac{M}{s-c}, \quad s > c.$$

Since the integral in $I_2$ converges absolutely for $s > c$, $I_2$ converges for $s > c$. Thus, both $I_1$ and $I_2$ exist and hence $I$ exists for $s > c$.

Comment The above condition is not necessary. For example, consider $f(t) = 1/\sqrt{t}$, which is not piecewise continuous in $[0, \infty)$. But
$$\int_0^\infty \frac{e^{-st}}{\sqrt{t}} \, dt = \frac{1}{\sqrt{s}} \int_0^\infty e^{-u}u^{1/2-1} \, du = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad s > 0.$$

3 Basic properties of Laplace transform

Theorem 1. (Uniqueness of Laplace transform) Let $f(t)$ and $g(t)$ be two functions such that $F(s) = G(s)$ for all $s > k$. Then $f(t) = g(t)$ at all $t$ where both are continuous.

Proposition 1. (Linearity) Suppose $F_1(s) = \mathcal{L}(f_1(t))$ exists for $s > a_1$ and $F_2(s) = \mathcal{L}(f_2(t))$ exists for $s > a_2$. Then
$$\mathcal{L}\left(c_1f_1(t) + c_2f_2(t)\right) = c_1F_1(s) + c_2F_2(s),$$
for $s > a$, where $a = \max\{a_1, a_2\}$.

Proof: Trivial

Example 9. Consider $f(t) = \cosh(\omega t)$. Then
$$\mathcal{L}\left(\cosh(\omega t)\right) = \frac{1}{2}\left(\mathcal{L}(e^{\omega t}) + \mathcal{L}(e^{-\omega t})\right) = \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) = \frac{s}{s^2 - \omega^2}.$$

Example 10. Consider $f(t) = \sinh(\omega t)$. Proceeding as above, we find
$$F(s) = \omega/(s^2 - \omega^2)$$
Theorem 2. (First shifting theorem) If \( \mathcal{L}(f(t)) = F(s) \), then

\[
\mathcal{L}(e^{at}f(t)) = F(s - a), \quad \text{and} \quad e^{at}f(t) = \mathcal{L}^{-1}\left(F(s - a)\right).
\]

**Proof:** Suppose \( \mathcal{L}(f(t)) = F(s) \) holds for \( s > k \). Now

\[
\mathcal{L}(e^{at}f(t)) = \int_{0}^{\infty} e^{-st}e^{at}f(t)\,dt = \int_{0}^{\infty} e^{-(s-a)t}f(t)\,dt = F(s - a), \quad s - a > k.
\]

**Example 11.** Consider \( f(t) = e^{-5t}\cos(4t) \). Since

\[
\mathcal{L}\left(\cos(4t)\right) = \frac{s}{s^2 + 16} \quad \Rightarrow \quad \mathcal{L}\left(e^{-5t}\cos(4t)\right) = \frac{s + 5}{(s + 5)^2 + 16}
\]

**Proposition 2.** If \( \mathcal{L}(f(t)) = F(s) \), then \( F(s) \to 0 \) as \( s \to \infty \).

**Proof:** We prove this for a piecewise continuous function which is of exponential order. But the result is valid for any function for which Laplace transform exists. Now

\[
I = \int_{0}^{\infty} e^{-st}f(t)\,dt \quad \Rightarrow \quad |I| \leq \int_{0}^{\infty} e^{-st}|f(t)|\,dt.
\]

Now since the function is exponential order, there exists \( M, \alpha, A \) such that \( |f(t)| \leq M_1e^{\alpha t} \) for \( t \geq A \). Also, since the function is piecewise continuous in \([0, A]\), we must have \( |f(t)| \leq M_2e^{\beta t} \) for \( 0 \leq t \leq A \) except possibly at some finite number of points where \( f(t) \) is not defined. Now we take \( M = \max\{M_1, M_2\} \) and \( \gamma = \max\{\alpha, \beta\} \). Then we have

\[
|F(s)| = |I| \leq \int_{0}^{\infty} e^{-st}|f(t)|\,dt \leq M \int_{0}^{\infty} e^{-(s-\gamma)t}\,dt = \frac{M}{s - \gamma}, \quad s > \gamma.
\]

Thus, \( F(s) \to 0 \) as \( s \to \infty \).

**Comment:** Any function \( F(s) \) without this behaviour can not be Laplace transform of a certain function. For example, \( s/(s-1) \), \( \sin s \), \( s^2/(1+s^2) \) are not Laplace transform of any function.