Lecture XIX
Laplace Transform of Periodic Functions, Convolution, Applications

1 Laplace transform of periodic function

**Theorem 1.** Suppose that \( f : [0, \infty) \to \mathbb{R} \) is a periodic function of period \( T > 0 \), i.e. \( f(t + T) = f(t) \) for all \( t \geq 0 \). If the Laplace transform of \( f \) exists, then

\[
F(s) = \int_0^T f(t)e^{-st} \, dt / (1 - e^{-sT}). \tag{1}
\]

**Proof:** We have

\[
F(s) = \int_0^\infty f(t)e^{-st} \, dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t)e^{-st} \, dt = \sum_{n=0}^{\infty} \int_0^T f(u+nT)e^{-su-snT} \, du = \sum_{n=0}^{\infty} e^{-snT} \int_0^T f(u)e^{-su} \, du = (\int_0^T f(u)e^{-su} \, du) \sum_{n=0}^{\infty} e^{-snT} = \int_0^T f(u)e^{-su} \, du / (1 - e^{-sT}).
\]

The last line follows from the fact that

\[
\sum_{n=0}^{\infty} e^{-snT}
\]

is a geometric series with common ration \( e^{-sT} < 1 \) for \( s > 0 \).

**Example 1.** Consider \( f(t) = \sin(\omega t) \), which is a periodic function of period \( 2\pi/\omega \).

**Solution:** Using (1), we find

\[
F(s) = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} \sin(\omega t) \, dt = \frac{\omega}{s^2 + \omega^2} \frac{1 - e^{-2\pi s/\omega}}{1 - e^{-2\pi s/\omega}} = \frac{\omega}{s^2 + \omega^2}
\]

**Example 2.** Consider a saw-tooth function (see Figure 1)

\[
f(t) = \begin{cases} t, & 0 \leq t < 1, \\ f(t-1), & t \geq 1. \end{cases}
\]
Solution: Here period $T = 1$. Using (1), we find
\[
F(s) = \frac{1}{1 - e^{-s}} \int_0^1 te^{-st} dt = \frac{1 - e^{-s}(1 + s)}{s^2(1 - e^{-s})} = \frac{1}{s^2} - \frac{e^{-s}}{s(1 - e^{-s})}
\]

Example 3. Consider the following function (see Figure 2)
\[
f(t) = \begin{cases} 
  t, & 0 \leq t < 1, \\
  2 - t, & 1 \leq t < 2, \\
  f(t - 2), & t \geq 2.
\end{cases}
\]

Solution: Here $f(t)$ is a periodic function of period $T = 2$. Hence, using (1), we find
\[
F(s) = \frac{1}{1 - e^{-2s}} \int_0^2 f(t)e^{-st} dt = \frac{1}{1 - e^{-2s}} \left( \int_0^1 te^{-st} dt + \int_1^2 (2 - t)e^{-st} dt \right)
\]
Simplifying the RHS, we find
\[
F(s) = \frac{(1 - e^{-s})^2}{s^2(1 - e^{-2s})} = \frac{1 - e^{-s}}{s^2(1 + e^{-s})} = \frac{1}{s^2} \tanh(s/2)
\]

Aliter: Note that
\[
f'(t) = \begin{cases} 
  1, & 0 < t < 1, \\
  -1, & 1 < t < 2, \\
  f(t - 2), & t > 2.
\end{cases}
\]
Since $f'$ is piecewise continuous and is of exponential order, its Laplace transform exist. Also, $f'$ is periodic with period $T = 2$. Hence,

$$\mathcal{L}(f') = \frac{1}{1 - e^{-2s}} \int_0^2 f'(t)e^{-st} \, dt = \frac{1}{1 - e^{-2s}} \left( \int_0^1 e^{-st} \, dt + \int_1^2 e^{-st} \, dt \right) = \frac{(1 - e^{-s})^2}{s(1 - e^{-2s})}.$$ 

Hence,

$$\mathcal{L}(f') = \frac{1}{s} \tanh(s/2) \implies sF(s) - f(0) = \frac{1}{s} \tanh(s/2) \implies F(s) = \frac{1}{s^2} \tanh(s/2).$$

**Comment:** Is it possible to do similar calculations (like in aliter) in Example 2? If not, why not?

## 2 Convolution

Suppose we know that a Laplace transform $H(s)$ can be written as $H(s) = F(s)G(s)$, where $\mathcal{L}(f(t)) = F(s)$ and $\mathcal{L}(g(t)) = G(s)$. We need to know the relation of $h(t) = \mathcal{L}^{-1}(H(s))$ to $f(t)$ and $g(t)$.

**Definition 1. (Convolution)** Let $f$ and $g$ be two functions defined in $[0, \infty)$. Then the convolution of $f$ and $g$, denoted by $f * g$, is defined by

$$ (f * g)(t) = \int_0^\infty f(\tau)g(t - \tau) \, d\tau \quad (2) $$

**Note:** It can be shown (easily) that $f * g = g * f$. Hence,

$$ (f * g)(t) = \int_0^\infty g(\tau)f(t - \tau) \, d\tau \quad (3) $$

We use either (2) or (3) depending on which is easier to evaluate.

**Theorem 2. (Convolution theorem)** The convolution $f * g$ has the Laplace transform property

$$ \mathcal{L}((f * g)(t)) = F(s)G(s). \quad (4) $$

**OR conversely**

$$ \mathcal{L}^{-1}(F(s)G(s)) = (f * g)(t) $$

**Proof:** Using definition, we find

$$ \mathcal{L}((f * g)(t)) = \int_0^\infty (f * g)(t)e^{-st} \, dt $$

$$ = \int_0^\infty \left( \int_0^t f(\tau)g(t - \tau) \, d\tau \right) e^{-st} \, dt $$

The region of integration is the area in the first quadrant bounded by the $t$-axis and
the line $\tau = t$. The variable limit of integration is applied on $\tau$ which varies from $\tau = 0$ to $\tau = t$.

Let us change the order of integration, thus apply variable limit on $t$. Then $t$ would vary from $t = \tau$ to $t = \infty$ and $\tau$ would vary from $\tau = 0$ to $\tau = \infty$. Hence, we have

$$L\left( (f * g)(t) \right) = \int_0^\infty \left( \int_\tau^\infty e^{-st} g(t - \tau) \, dt \right) f(\tau) \, d\tau$$

$$= \int_0^\infty \left( \int_0^\infty e^{-su} g(u) \, du \right) f(\tau) e^{-s\tau} \, d\tau, \quad t - \tau = u$$

$$= \left( \int_0^\infty e^{-su} g(u) \, du \right) \left( \int_0^\infty e^{-s\tau} f(\tau) \, d\tau \right)$$

$$= F(s)G(s)$$

**Example 4.** Consider the same problem as given in Example 4 of Lecture Note 18, i.e. find inverse Laplace transform of $1/s(s + 1)^2$.

**Solution:** We write $H(s) = F(s)G(s)$, where $F(s) = 1/s$ and $G(s) = 1/(s + 1)^2$. Thus $f(t) = 1$ and $g(t) = te^{-t}$. Hence, using convolution theorem, we find

$$h(t) = \int_0^t (t - \tau)g(\tau) \, d\tau = \int_0^t \tau e^{-\tau} \, d\tau = 1 - (t + 1)e^{-t}.$$  

**Note:** We have used $f(t - \tau)g(\tau)$ in the convolution formula since $f(t) = 1$. This helps a little bit in the evaluation of the integration.

**Example 5.** Find inverse Laplace transform of $1/(s^2 + \omega^2)$.

**Solution:** Let $H(s) = F(s)G(s)$, where $F(s) = 1/(s^2 + \omega^2)$ and $G(s) = 1/(s^2 + \omega^2)$. 
Thus, \( f(t) = \sin(\omega t)/\omega = g(t) \). Hence,

\[
h(t) = \frac{1}{\omega^2} \int_0^t \sin(\omega \tau) \sin(\omega (t - \tau)) d\tau = \frac{1}{2\omega^3} \left( \sin(\omega t) - \omega t \cos(\omega t) \right).
\]

### 3 Applications

**Example 6.** *(Differential equation)* Solve the IVP

\[
y'' + y = t, \quad y(0) = 0, \quad y'(0) = 2
\]

**Solution:** Take Laplace transform on both sides. This gives

\[
s^2 Y - 2 + Y = \frac{1}{s^2} \Rightarrow Y = \frac{1}{s^2} + \frac{2}{s^2 + 1}
\]

Using partial fraction, we find

\[
Y = \frac{1}{s^2} + \frac{1}{s^2 + 1} \Rightarrow y(t) = t + \sin t
\]

**Aliter:** In the method above, we evaluated Laplace transform of the nonhomogeneous term in the right hand side. Now here we don’t evaluate it. Let \( g(t) \) be nonhomogeneous term (in this case \( g(t) = t \)). Let \( G(s) \) be the Laplace transform of \( g \). Now Take Laplace transform on both sides. This gives

\[
s^2 Y - 2 + Y = G(s) \Rightarrow Y = \frac{G(s)}{s^2} + \frac{2}{s^2 + 1}
\]

Taking inverse transform and convolution, we find

\[
y(t) = \int_0^t g(t - \tau) \sin(\tau) d\tau + 2 \sin t \Rightarrow y(t) = \int_0^t (t - \tau) \sin(\tau) d\tau + 2 \sin t
\]

OR (using integration by parts)

\[
y(t) = t + \sin t
\]

**Example 7.** *(Differential equation)* Solve the IVP

\[
y'' + 9y = \begin{cases} 8\sin t, & 0 < t < \pi, \\ 0, & t > \pi, \end{cases} \quad y(0) = 0, \quad y'(0) = 4.
\]

**Solution:** Consider \( g(t) = 8\left( u_0(t) - u_\pi(t) \right) \sin t \). Then Laplace transform of the nonhomogeneous term is the same as that of \( g(t) \). Now we write \( g(t) \) as

\[
g(t) = 8u_0(t) \sin t + 8u_\pi(t) \sin(t - \pi).
\]
Now taking Laplace transform of the ODE, we get
\[ s^2 Y - 4 + 9Y = \frac{8}{s^2 + 1} + 8e^{-\pi s} \]
\[ \implies Y = \frac{4}{s^2 + 9} + 8 \frac{1}{(s^2 + 1)(s^2 + 9)} + 8e^{-\pi s} \frac{1}{(s^2 + 1)(s^2 + 9)} \]

Using partial fraction, we get
\[ Y = \frac{4}{s^2 + 9} + \left( \frac{1}{s^2 + 1} - \frac{1}{s^2 + 9} \right) + e^{-\pi s} \left( \frac{1}{s^2 + 1} - \frac{1}{s^2 + 9} \right). \]

Now
\[ \mathcal{L} \left( \sin t - \frac{1}{3} \sin 3t \right) = \frac{1}{s^2 + 1} - \frac{1}{s^2 + 9}. \]

Hence, using shifting theorem and inverse transform, we find
\[ y(t) = \frac{4}{3} \sin 3t + \sin t - \frac{1}{3} \sin 3t + u_\pi(t) \left( \sin(t - \pi) - \frac{1}{3} \sin 3(t - \pi) \right) \]

Further, this can be break up as
\[ y(t) = \begin{cases} 
\sin 3t + \sin t, & 0 \leq t \leq \pi, \\
\frac{4}{3} \sin 3t, & t \geq \pi, 
\end{cases} \]

**Example 8. (Differential equation) (Variable coefficient)** Solve the IVP

\[ y'' - 2xy' + 4y = 0, \quad y(0) = 1, y'(0) = 0 \]

**Solution:** Take Laplace Transform on both sides, we find
\[ s^2 Y - sy(0) - y'(0) + 2 \frac{d}{ds} \left( \mathcal{L}(y') \right) + 4Y = 0, \]

OR
\[ s^2 Y - s + 2 \frac{d}{ds} \left( sY - y(0) \right) + 4Y = 0 \implies 2sY' + (s^2 + 6)Y = s \implies Y' + \left( \frac{s}{2} + \frac{3}{s} \right)Y = \frac{1}{2} \]

This is linear equation. Hence,
\[ Y s^3 e^{s^2/4} = \frac{1}{2} \int s^3 e^{s^2/4} ds + C \]

OR
\[ Y = \frac{s^2 - 4}{s^3} + C \frac{e^{-s^2/4}}{s^3} \]

OR
\[ Y = \frac{1}{s} - \frac{4}{s^3} + C \frac{e^{-s^2/4}}{s^3}. \]

Now it can be shown by Bromwich integral method (not in the syllabus) that
\[ \mathcal{L} \left( \frac{x^2}{2} - \frac{1}{4} \right) = \frac{e^{-s^2/4}}{s^3} \]
Hence, we find
\[ y(t) = (1 - 2x^2) + C \left( \frac{x^2}{2} - \frac{1}{4} \right). \]

OR
\[ y(t) = (1 - C/4) + (C/2 - 2)x^2 \]

Now \( y(0) = 1 \implies C = 4. \) Hence
\[ y(x) = (1 - 2x^2) \]

Comment: If we expand \( e^{-s^2/4}/s^3 \) then we find
\[ \frac{e^{-s^2/4}}{s^3} = \frac{1}{s^3} - \frac{1}{4s} + \text{non-negative power of } s. \]

If we assume \( \mathcal{L}^{-1}(s^k) = 0, \ k = 0, 1, 2, \cdots, \) then we find
\[ \mathcal{L} \left( \frac{x^2}{2} - \frac{1}{4} \right) = \frac{e^{-s^2/4}}{s^3} \]

Example 9. (Integral equation) Solve
\[ y' + \int_0^t y(t - \tau)e^{-2\tau}d\tau = 1, \quad y(0) = 1. \]

Solution: Take Laplace Transform on both sides, we find
\[ sY - y(0) + \frac{Y}{s + 2} = \frac{1}{s} \implies Y = \frac{s + 2}{s(s + 1)} \implies Y = \frac{2}{s} - \frac{1}{s + 1} \]

Hence,
\[ y(t) = 2 - e^{-t} \]