Lecture III
Solution of first order equations

1 Separable equations

These are equations of the form

\[ y' = f(x)g(y) \]

Assuming \( g \) is nonzero, we divide by \( g \) and integrate to find

\[ \int \frac{dy}{g(y)} = \int f(x)dx + C \]

What happens if \( g(y) \) becomes zero at a point \( y = y_0 \)?

Example 1. \( xy' = y + y^2 \)

Solution: We write this as

\[ \int \frac{dy}{y + y^2} = \int \frac{dx}{x} + C \Rightarrow \int \frac{dy}{y} - \int \frac{dy}{1 + y} = \ln x + C \Rightarrow \ln y - \ln(1 + y) = \ln x + C \]

Note: Strictly speaking, we should write the above solution as

\[ \ln |y| - \ln |1 + y| = \ln |x| + C \]

When we wrote the solution without the modulus sign, it was (implicitly) assumed that \( x > 0, y > 0 \). This is acceptable for problems in which the solution domain is not given explicitly. But for some problems, the modulus sign is necessary. For example, consider the following IVP:

\[ xy' = y + y^2, \quad y(-1) = -2. \]

Try to solve this.

2 Reduction to separable form

2.1 Substitution method

Let the ODE be

\[ y' = F(ax + by + c) \]

Suppose \( b \neq 0 \). Substituting \( ax + by + c = v \) reduces the equation to a separable form. If \( b = 0 \), then it is already in separable form.

Example 2. \( y' = (x + y)^2 \)

Solution: Let \( v = x + y \). Then we find

\[ v' = v^2 + 1 \Rightarrow \tan^{-1} v = x + C \Rightarrow x + y = \tan(x + C) \]
2.2 Homogeneous form

Let the ODE be of the form

\[ y' = f(y/x) \]

In this case, substitution of \( v = y/x \) reduces the above ODE to a separable ODE.

Comment 1: Sometimes, substitution reduces an ODE to the homogeneous form. For example, if \( ae \neq bd \), then \( h \) and \( k \) can be chosen so that \( x = u + h \) and \( y = v + k \) reduces the following ODE

\[ y' = F\left(\frac{ax + by + c}{dx + ey + f}\right) \]

to a homogeneous ODE. What happens if \( ae = bd \)?

Comment 2: Also, an ODE of the form

\[ y' = y/x + g(x)h(y/x) \]

can be reduced to the separable form by substituting \( v = y/x \).

Example 3. \( xyy' = y^2 + 2x^2, \quad y(1) = 2 \)

Solution: Substituting \( v = y/x \) we find

\[ v + xv' = v + 2/v \Rightarrow y^2 = 2x^2(C + \ln x^2) \]

Using \( y(1) = 2 \), we find \( C = 2 \). Hence, \( y = 2x^2(1 + \ln x^2) \)

3 Exact equation

A first order ODE of the form

\[ M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1} \]

is exact if there exits a function \( u(x, y) \) such that

\[ M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}. \]

Then the above ODE can be written as \( du = 0 \) and hence the solution becomes \( u = C \).

**Theorem 1.** Let \( M \) and \( N \) be defined and continuously differentiable on a rectangle \( R = \{(x, y) : |x - x_0| < a, |y - y_0| < b\} \). Then (1) is exact if and only if \( \partial M/\partial y = \partial N/\partial x \) for all \( (x, y) \in R \).

**Proof:** We shall only prove the necessary part. Assume that (1) is exact. Then there exits a function \( u(x, y) \) such that

\[ M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}. \]

Since \( M \) and \( N \) have continuous first partial derivatives, we have

\[ \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}. \]

Now continuity of 2nd partial derivative implies \( \partial M/\partial y = \partial N/\partial x \).
Example 4. Solve \((2x + \sin x \tan y)dx - \cos x \sec^2 y dy = 0\)

Solution: Here \(M = 2x + \sin x \tan y\) and \(N = -\cos x \sec^2 y\). Hence, \(M_y = N_x\). Hence, the solution is \(u = C\), where \(u = x^2 - \cos x \tan y\)

4 Reduction to exact equation: integrating factor

An integrating factor \(\mu(x, y)\) is a function such that

\[ M(x, y) \, dx + N(x, y) \, dy = 0 \]  \hspace{1cm} (2)

becomes exact on multiplying it by \(\mu\). Thus,

\[ \mu M \, dx + \mu N \, dy = 0 \]

is exact. Hence

\[ \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \]

Comment: If an equation has an integrating factor, then it has infinitely many integrating factors.

Proof: Let \(\mu\) be an integrating factor. Then

\[ \mu M \, dx + \mu N \, dy = du \]

Let \(g(u)\) be any continuous function of \(u\). Now multiplying by \(\mu g(u)\), we find

\[ \mu g(u)M \, dx + \mu g(u)N \, dy = g(u)du \Rightarrow \mu g(u)M \, dx + \mu g(u)N \, dy = d\left(\int g(u) \, du\right) \]

Thus,

\[ \mu g(u)M \, dx + \mu g(u)N \, dy = dv, \quad \text{where} \quad v = \int g(u) \, du \]

Hence, \(\mu g(u)\) is an integrating factor. Since, \(g\) is arbitrary, there exists an infinite number of integrating factors.

Example 5. \(xdy - ydx = 0\).

Solution: Clearly \(1/x^2\) is an integrating factor since

\[ \frac{xdy - ydx}{x^2} = 0 \Rightarrow d(y/x) = 0 \]

Also, \(1/xy\) is an integrating factor since

\[ \frac{xdy - ydx}{xy} = 0 \Rightarrow d\ln(y/x) = 0 \]

Similarly it can be shown that \(1/y^2, 1/(x^2 + y^2)\) etc. are integrating factors.
4.1 How to find integrating factor

Theorem 2. If (2) is such that
\[ \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \]
is a function of x alone, say $F(x)$, then
\[ \mu = e^{\int F \, dx} \]
is a function of x only and is an integrating factor for (2).

Example 6. $(xy - 1)\, dx + (x^2 - xy)\, dy = 0$

Solution: Here $M = xy - 1$ and $N = x^2 - xy$. Also,
\[ \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = -\frac{1}{x} \]
Hence, $1/x$ is an integrating factor. Multiplying by $1/x$ we find
\[ \frac{(xy - 1)\, dx + (x^2 - xy)\, dy}{x} = 0 \Rightarrow xy - \ln x - y^2/2 = C \]

Theorem 3. If (2) is such that
\[ -\frac{1}{M} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \]
is a function of y alone, say $G(y)$, then
\[ \mu = e^{\int G \, dy} \]
is a function of y only and is an integrating factor for (2).

Example 7. $y^3 \, dx + (xy^2 - 1)\, dy = 0$

Solution: Here $M = y^3$ and $N = xy^2 - 1$. Also,
\[ -\frac{1}{M} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = -\frac{2}{y} \]
Hence, $1/y^2$ is an integrating factor. Multiplying by $1/y^2$ we find
\[ \frac{y^3\, dx + (xy^2 - 1)\, dy}{y^2} = 0 \Rightarrow xy + \frac{1}{y} = C \]

Comment: Sometimes it may be possible to find integrating factor by inspection. For this, some known differential formulas are useful. Few of these are given below:

\[ d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2} \]
\[ d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \]
\[ d(xy) = xdy + ydx \]
\[ d\left(\ln\frac{x}{y}\right) = \frac{ydx - xdy}{xy} \]
Example 8. \((2x^2y + y)dx + xdy = 0\)

Obviously, we can write this as

\[
2x^2ydx + (ydx + xdy) = 0 \Rightarrow 2x^2ydx + d(xy) = 0
\]

Now if we divide this by \(xy\), then the last term remains differential and the first term also becomes differential:

\[
2xdx + \frac{d(xy)}{xy} = 0 \Rightarrow d\left(x^2 + \ln(xy)\right) = 0 \Rightarrow x^2 + \ln(xy) = C
\]