Viscous Fluid

1 Description of motion

Let $P$ be a fluid particle which at time $t_0$ occupied the position $X$ and at time $t$ occupies the position $x$. Since each fluid particle have unique positions at $t_0$ and $t$, we can define a mapping $\kappa$ such that

$$X = \kappa(P, t_0) \quad \text{and} \quad x = \kappa(P, t)$$

The location at $t_0$ is known as reference coordinate and at $t$ is known as current coordinate. We can eliminate $P$ from the above to get

$$x = x(X, t) \quad \text{and} \quad X = X(x, t)$$

The problem in fluid mechanics may be formulated either using $(X, t)$ as independent variables (material or Lagrangian description) or using $(x, t)$ as independent variables (spatial or Eulerian description).

2 Material derivative

The time rate change of a quantity (scalar, vector or tensor) of a material particle is known as material time derivative. There are two cases to consider.

(i) The material description of the quantity is used i.e. the quantity is a function of $X$ and $t$. We only have to take partial derivative with respect to $t$. Thus if $\phi(X, t)$ is a scalar field, then its material derivative is given by

$$D\phi \frac{D}{Dt} := \frac{\partial \phi(X, t)}{\partial t}$$

(ii) The spatial description of the quantity is used. If $\phi(x, t)$ is a scalar field, then its material derivative is given by

$$D\phi \frac{D}{Dt} := \left( \frac{\partial \phi(x(X, t), t)}{\partial t} \right)_{x-\text{fixed}}$$

This could be simplified as follows: Since

$$\phi(x(X, t), t) = \phi(x_1(X, t), x_2(X, t), x_3(X, t), t)$$
we have
\[
\left( \frac{\partial \phi(x(X, t), t)}{\partial t} \right)_{x\text{-fixed}} = \frac{\partial \phi(x(X, t), t)}{\partial x_1} \frac{\partial x_1(X, t)}{\partial t} + \frac{\partial \phi(x(X, t), t)}{\partial x_2} \frac{\partial x_2(X, t)}{\partial t} + \frac{\partial \phi(x(X, t), t)}{\partial x_3} \frac{\partial x_3(X, t)}{\partial t}
\]
Now we come back to the spatial coordinates by substituting \( X = X(x, t) \) to get
\[
\left. \frac{d\phi(x(X, t), t)}{dt} \right|_{X = X(x, t)} = \frac{\partial \phi(x, t)}{\partial x_i} v_i(x, t) + \frac{\partial \phi(x, t)}{\partial t} = \mathbf{v} \cdot (\nabla \phi) + \frac{\partial \phi(x, t)}{\partial t}
\]
where \( \mathbf{v} \) is the Eulerian velocity field. Thus for Eulerian field we have important formula
\[
\frac{D\phi}{Dt} := \mathbf{v} \cdot (\nabla \phi) + \frac{\partial \phi(x, t)}{\partial t}
\]
Next let \( \mathbf{u}(x, t) \) be an Eulerian vector field. So we have
\[
\frac{D\mathbf{u}}{Dt} := \left( \frac{\partial \mathbf{u}(x(X, t), t)}{\partial t} \right)_{x\text{-fixed}}
\]
Now the components \( u_i(x, t) \) of \( \mathbf{u}(x, t) \) is a scaler field. Hence using the above result we write
\[
\frac{Du_i}{Dt} := \frac{\partial u_i(x, t)}{\partial t} + \mathbf{v} \cdot (\nabla u_i) = \frac{\partial u_i(x, t)}{\partial t} + (\nabla \mathbf{u})_{ij} v_j
\]
Thus
\[
\frac{D\mathbf{u}}{Dt} := \frac{\partial \mathbf{u}(x, t)}{\partial t} + (\nabla \mathbf{u}(x, t)) \mathbf{v}(x, t)
\]
3 Deformation gradient and change of area and volume element

From
\[
x = x(X, t)
\]
we can write
\[
dx = FdX
\]
where \( F \) is known as deformation gradient.
Let \((dA, N)\) and \((da, n)\) be the elemental surface and normal pairs in the reference and current frame respectively. Then we have

\[
n da = JF^{-T}N dA
\]

where \(J = det(F)\). Also if \(dv\) and \(dV\) are the volume in the current and reference configuration respectively, then

\[
dv = J dV
\]

4 Velocity gradient

The velocity gradient \(L\) is defined as the spatial gradient of the velocity i.e.

\[
L = \nabla v, \quad L_{ij} = v_{i,j}
\]

\(L\) can be decompose as a sum of symmetric and skew-symmetric part i.e.

\[
L = D + W
\]

where

\[
D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad W_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})
\]

Some useful relations involving velocity gradient are

\[
\frac{DF}{Dt} = LF
\]

\[
\frac{DJ}{Dt} = Jtr(L) = Jv_{i,i}
\]

5 Transport formulas

We are interested in calculating the rate of change of integrals over material curves, surfaces and volumes respectively. Let \(C_t, S_t\) and \(\Omega_t\) denote the material curve, surface and volume respectively in the current frame. Then we have the following
formulas

\[
\frac{d}{dt} \int_{C_t} \phi dx = \int_{C_t} \left( \frac{D\phi}{Dt} + \phi L \right) dx
\]

\[
\frac{d}{dt} \int_{S_t} \phi n da = \int_{S_t} \left[ \left( \frac{D\phi}{Dt} + \phi \text{tr}(L) \right) n - \phi L^T n \right] da
\]

\[
\frac{d}{dt} \int_{\Omega_t} \phi dv = \int_{\Omega_t} \left( \frac{D\phi}{Dt} + \phi \text{tr}(L) \right) dv
\]

\[
\frac{d}{dt} \int_{C_t} u \cdot dx = \int_{C_t} \left( \frac{Du}{Dt} + L^T u \right) \cdot dx
\]

\[
\frac{d}{dt} \int_{S_t} u \cdot n da = \int_{S_t} \left( \frac{Du}{Dt} + u \text{tr}(L) - Lu \right) \cdot n da
\]

\[
\frac{d}{dt} \int_{\Omega_t} udv = \int_{\Omega_t} \left( \frac{Du}{Dt} + u \text{tr}(L) \right) dv
\]

6 Streamlines, stream tubes, path lines and streak lines

Streamlines are lines whose tangents are everywhere parallel to the velocity vector. Since in unsteady flow, the velocity vectors change both magnitude and direction with time, it is meaningful to consider only the instanteneous streamlines in the case of unsteady flow. A streamline can not cross other streamlines except at the stagnation point. The equation of streamline can be written as

\[
\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}
\]

We introduce a parameter \( s \) whose value is zero at some reference point and whose value increases along the streamline. Thus we have

\[
\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = ds
\]

or

\[
\frac{dx}{ds} = v(x, t), \quad t \text{ fixed}
\]

The collection of all streamlines that pass through an open line in a flow forms a stream surface and the collection of all streamlines that pass through a closed loop forms a stream tube. Let us consider two closed loops that wrap around a particular stream tube. Volumetric flow rate through \( S_t \) is given by
\[ Q_i = \int_{S_i} \mathbf{v} \cdot \mathbf{n} \, dS \]

Now we have
\[ Q_2 - Q_1 = \int_S \mathbf{v} \cdot \mathbf{n} \, dS + \int_{V_1} \nabla \cdot \mathbf{v} \, dV = \int_{V_1} \nabla \cdot \mathbf{v} \, dV \]

Volumetric flow rate may increase or decrease according to whether the fluid inside the tube is undergoing expansion or contraction.

A pathline is a line traced out in time by a given fluid particle as it flows. Since the particle under consideration is moving with the fluid at its local velocity, pathlines must satisfy the equations
\[ \frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) \]

The equation of the pathline that passes through the point \( \mathbf{X} \) at time \( t_0 \) will then be the solution of the above equation which satisfies the initial condition \( \mathbf{x}(t = t_0) = \mathbf{X} \).

A streakline is the locus of all fluid particles that have passed through a prescribed fixed point during a specific interval of time. An example of a streakline is a line traced by the continuous injection at a certain point of dye, smoke, or bubbles. In the case of pathline, we have time \( t \) varying from \( t = t_0 \) to \( t = \infty \). In the case of streakline, the time \( t \) is fixed and \( t_0 \) is allowed to have all admissable values.

**Example:** Consider the two-dimensional flow field defined by
\[ u = x(1 + 2t) \]
\[ v = y \]
\[ w = 0 \]

The streamline which passes through \((1, 1)\) in parametric form is
\[ x = e^{(1+2t)s} \]
\[ y = e^s \]
The streamline passing through \((1, 1)\) at \(t = 0\) is
\[
\begin{align*}
x &= e^s \\
y &= e^s
\end{align*}
\]
or eliminating \(s\) we get
\[x = y\]
The pathline of the particles passing through \((1, 1)\) at \(t = 0\) is
\[
\begin{align*}
x &= e^{(1+t)t} \\
y &= e^t
\end{align*}
\]
or eliminating \(t\) we get
\[x = y^{1+\log y}\]
The pathline of the particle which is at \((1, 1)\) at \(t = \tau\) is given by
\[
\begin{align*}
x &= e^{(1+t)t-\tau(1+\tau)} \\
y &= e^{(t - \tau)}
\end{align*}
\]
These are parametric equations for streakline that passes through \((1, 1)\) and are valid for all times \(t\). In particular the streakline at \(t = 0\) in parametric form is
\[
\begin{align*}
x &= e^{-\tau(1+\tau)} \\
y &= e^{-\tau}
\end{align*}
\]
or eliminating \(\tau\) we get
\[x = y^{1-\log y}\]
Thus none of the three flow lines coincide. But if the flow is steady, then they must coincide.

7 Vorticity

We know that the tensor \(\mathbf{W}\) defined by
\[
\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) \quad \text{or} \quad W_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})
\]
is skew symmetric. Hence its dual vector is given by \(\mathbf{w} = (w_1, w_2, w_3)\) where
\[
w_i = \frac{1}{2} \epsilon_{ijk} v_{k,j}
\]
The vorticity $\omega$ is twice the dual vector of $W$. Hence

$$\omega_i = \epsilon_{ijk} v_{k,j} \quad \text{or} \quad \omega = \nabla \wedge v$$

It is sometimes useful to understand the dynamics of flow by working with the vorticity rather than the velocity field. Thus we would like to have equation for $\omega$. The following derivations are based on the incompressible fluid with $\nu = \mu/\rho$. The constant $\nu$ is called kinematic viscosity. We have in absence of body forces

$$\frac{\partial u}{\partial t} + (\nabla u)u = -\nabla(p/\rho) + \nu \nabla^2 u$$

Now we use the following two identities

$$\nabla^2 u = \nabla(\nabla \cdot u) - \nabla \wedge \omega$$

$$(\nabla u)u = \nabla \left( \frac{|u|^2}{2} \right) + \omega \wedge u$$

Thus we obtained

$$\frac{\partial u}{\partial t} + \omega \wedge u = -\nabla \left( \frac{|u|^2}{2} + \frac{p}{\rho} \right) - \nu \nabla \wedge \omega$$

Again taking curl on both sides we get

$$\frac{\partial \omega}{\partial t} + \nabla \wedge (\omega \wedge u) = -\nu \nabla \wedge (\nabla \wedge \omega)$$

$$= \nu \nabla^2 \omega$$

Again using the identity

$$\nabla \wedge (a \wedge b) = (\nabla a) b - (\nabla b) a + (\nabla \cdot b) a - (\nabla \cdot a) b$$

we get

$$\frac{\partial \omega}{\partial t} + (\nabla \omega) u = \nu \nabla^2 \omega + (\nabla u) \omega$$

The term $\partial \omega/\partial t$ represents convection of vorticity with the flow, $\nu \nabla^2 \omega$ represent diffusion of vorticity and $(\nabla u) \omega$ represents stretching of the vortex line.

### 7.1 Vortex lines and vortex tubes

Just as a streamlines is a curve to which the velocity vector is tangent everywhere, we can define a vortex line is a curve in which the vorticity is tangent everywhere. Hence equation is given by

$$\frac{dx}{\omega_1} = \frac{dy}{\omega_2} = \frac{dz}{\omega_3}$$
A vortex tube is the series of vortex line passing through a closed curve. Consider a stream tube with one section of area $S_1$ and the other $S_2$. Since $\nabla \cdot \mathbf{\omega} = 0$ and if

$$\Gamma_i = \int_{S_i} \mathbf{\omega} \cdot \mathbf{n} \, dS, \quad i = 1, 2$$

then $\Gamma_1 = \Gamma_2$. Also using Stoke’s theorem we have

$$\Gamma_i = \int_{S_i} \mathbf{\omega} \cdot \mathbf{n} \, dS = \int_{C_i} \mathbf{u} \cdot d\mathbf{x}$$

where $C_i$ is the curve enclosing $S_i$. The last integral is the circulation along the curve.

### 7.2 Kelvin’s circulation theorem

We can also prove the circulation along a closed curve remains constant for an ideal barotropic fluid with conservative body force. For barotropic fluid we have $p = p(\rho)$ and hence if we define

$$P = \int p \, \frac{dp}{\rho} \quad \text{then} \quad \nabla P = \frac{\nabla p}{\rho}$$

Also for conservative body force we write $\mathbf{b} = \nabla \chi$. Thus We have

$$\Gamma = \int_{C_1} \mathbf{v} \cdot d\mathbf{x}$$

Using Reynolds circulation theorem we get

\[
\frac{d\Gamma}{dt} = \int_{C_1} \left[ \frac{Dv}{Dt} + (\nabla \mathbf{v})^T \mathbf{v} \right] \cdot d\mathbf{x} = \int_{C_1} \left[ -\frac{1}{\rho} \nabla p + \nabla \chi + (\nabla \mathbf{v})^T \mathbf{v} \right] \cdot d\mathbf{x} = \int_{C_1} \nabla \left( P + \chi + \frac{v_i v_i}{2} \right) \cdot d\mathbf{x} = \int_{C_1} d \left( P + \chi + \frac{v_i v_i}{2} \right) = 0
\]

### 8 The Stream function

Sometimes it is convenient to describe fluid flow in terms of a stream function, usually represented by $\psi$, that remains constant along stream line. The equation of mass conservation states that

$$\nabla \cdot \mathbf{v} = 0$$
We can take \( \mathbf{v} = \nabla \wedge \mathbf{A} \) as a general solution for this. However \( \mathbf{A} \) is not unique since if \( \mathbf{A}' = \mathbf{A} + \nabla q \) then

\[
\nabla \wedge \mathbf{a} = \nabla \wedge \mathbf{A}'
\]

Now the vorticity is

\[
\omega = \nabla \wedge \mathbf{v} = \nabla \wedge (\nabla \wedge \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla (\nabla \cdot \mathbf{A})
\]

Since \( \mathbf{A} \) is non unique, we impose the condition that \( \nabla \cdot \mathbf{v} = 0 \) and thus

\[
\omega = -\nabla^2 \mathbf{A}
\]

Examples

(i) Two dimensional flow: Here \( \mathbf{v} = u(x,y)\mathbf{i} + b(x,y)\mathbf{j} \). If we pick \( \mathbf{A} = \psi(x,y)\mathbf{k} \) then \( \nabla \cdot \mathbf{A} = \partial \psi / \partial z = 0 \). Now

\[
\mathbf{v} = \nabla \wedge (\psi \mathbf{k}) = \frac{\partial \psi}{\partial y} \mathbf{i} - \frac{\partial \psi}{\partial x} \mathbf{j}
\]

Hence

\[
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}
\]

For two dimensional flow we have \( \omega = -\nabla^2 \mathbf{A} = -\nabla^2 \psi \mathbf{k} \). Hence \( (\nabla \mathbf{v}) \omega = 0 \). Thus the vorticity equation becomes

\[
\frac{\partial \nabla^2 \psi}{\partial t} + (\nabla \nabla^2 \psi) \mathbf{v} = \nu \nabla^4 \psi
\]

Thus the Navier-Stokes equation can be written as

\[
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}
\]

\[
\frac{\partial \zeta}{\partial t} + (\nabla \zeta) \mathbf{v} = \nu \nabla^2 \zeta
\]

(ii) Axisymmetric flow: In cylindrical polar coordinate, the velocity is

\[
\mathbf{v} = u(r,z)\mathbf{e}_r + w(r,z)\mathbf{e}_z
\]

Let us choose \( \mathbf{A} = (\psi/r)\mathbf{e}_\theta \) where \( \psi = \psi(r,z) \). Now

\[
\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (rA_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} = \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} = 0
\]
Thus we can represent fluid velocity as

\[ \mathbf{v} = \nabla \wedge \mathbf{A} = \left( \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \mathbf{e}_r + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \mathbf{e}_\theta + \frac{1}{r} \left( \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \mathbf{e}_z \]

Thus we get

\[ u = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad w = \frac{1}{r} \frac{\partial \psi}{\partial r} \]

Alternatively if we have defined \( \mathbf{A} = \psi \mathbf{e}_\theta \), where \( \psi = \psi(r, z) \), then we obtain

\[ u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{1}{r} \frac{\partial}{\partial r} (r \psi) \]

Thus the stream function can be defined in the equivalent but different ways (ii)

*Spherical flow independent of azimuth \( \phi \):* Here we have

\[ \mathbf{v} = u(r, \theta) \mathbf{e}_r + w(r, \theta) \mathbf{e}_\theta \]

Let us choose

\[ \mathbf{A} = \frac{\psi(r, \theta)}{r \sin \theta} \mathbf{e}_\phi \]

then

\[ \nabla \cdot \mathbf{A} = \frac{1}{r^2 \sin^2 \theta} \frac{\partial \psi(r, \theta)}{\partial \phi} = 0 \]

Now we have for velocity

\[ \mathbf{v} = \nabla \wedge \mathbf{A} = \frac{1}{r^2 \sin \theta} \left( \frac{\partial \psi}{\partial \theta} \mathbf{e}_r - r \frac{\partial \psi}{\partial r} \mathbf{e}_\theta \right) \]

Thus componentwise

\[ u = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad w = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \]

### 8.1 Physical interpretation

For simplicity we consider two dimensional flow. In this case we have \( \mathbf{v} = \nabla \wedge (\psi \mathbf{k}) \)

Now using the identity

\[ \nabla \wedge (\phi \mathbf{a}) = \phi \nabla \wedge \mathbf{a} + \nabla \phi \wedge \mathbf{a} \]
we get
\[ \mathbf{v} \cdot \nabla \psi = \nabla \wedge (\psi \mathbf{k}) \cdot \nabla \psi = (\nabla \psi \wedge \mathbf{k}) \cdot \nabla \psi = \mathbf{k} \cdot (\nabla \psi \wedge \nabla \psi) = 0 \]

Thus the gradient of \( \psi \) along \( \mathbf{v} \) is zero. If \( dx \) is along the tangent to the streamline, then \( dx \) is proportional \( \mathbf{v} \) and hence along the streamline we have
\[ \mathbf{v} \cdot \nabla \psi = 0 \implies d\psi = 0 \]

Hence \( \psi \) is constant along the streamline. Let us consider two neighbouring streamlines with value of \( \psi \) on them as \( \psi \) and \( \psi + d\psi \) respectively. The unit vector in the direction shown is
\[ \mathbf{n} = (dy/ds, -dx/ds) \]

Now the flux of fluid between them in unit time is
\[ \mathbf{v} \cdot \mathbf{n}ds = u dy - v dx = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi \]

Thus the difference in \( \psi \) measures the flux per unit time.

9 The Stress tensor

The stress vector acting on the plane with unit normal \( \mathbf{n} \) is
\[ \mathbf{t}_n = \sigma^T \mathbf{n} \]

where \( \sigma \) is a second order tensor called stress tensor. The components \( \sigma_{11}, \sigma_{22}, \sigma_{33} \) are called normal stresses and \( \sigma_{ij}, i \neq j \) are shear stresses.
9.1 Stress tensor symmetric

Let $P_t \subset B_t$ be an arbitrary material region in the current configuration. The linear momentum $\mathcal{M}(P_t)$ of the material occupying the region $P_t$ in the current configuration is defined by

$$\mathcal{M}(P_t) = \int_{P_t} \rho v \, dv$$

If $\mathbf{x}$ is the position vector of a point in $P_t$ relative origin $\mathbf{o}$, then the angular momentum of $P_t$ with respect to $\mathbf{o}$ is defined by

$$\mathcal{G}(P_t; \mathbf{o}) = \int_{P_t} \mathbf{x} \wedge (\rho v) \, dv$$

Now applying the conservation of linear momentum for the material in $P_t$ we get

$$\frac{d}{dt} \int_{P_t} \rho v \, dv = \int_{P_t} \rho b \, dv + \int_{\partial P_t} \mathbf{t}(n) \, da,$$

and applying the conservation of angular momentum we get

$$\frac{d}{dt} \int_{P_t} \mathbf{x} \wedge (\rho v) \, dv = \int_{P_t} \mathbf{x} \wedge (\rho b) \, dv + \int_{\partial P_t} \mathbf{x} \wedge \mathbf{t}(n) \, da,$$

Using the result

$$\frac{d}{dt} \int_{P_t} \rho \phi \, dv = \int_{P_t} \rho \frac{D\phi}{Dt} \, dv$$

we get

$$\int_{P_t} \rho (a - b) \, dv = \int_{\partial P_t} \mathbf{t}(n) \, da$$

and

$$\int_{P_t} \rho \mathbf{x} \wedge (a - b) \, dv = \int_{\partial P_t} \mathbf{x} \wedge \mathbf{t}(n) \, da,$$

where $\mathbf{a} = \frac{D\mathbf{v}}{Dt}$ is the acceleration.

From the first relation we get

$$\int_{P_t} \rho (a - b) \, dv = \int_{\partial P_t} \mathbf{\sigma}^T \mathbf{n} \, da = \int_{\partial P_t} \text{div} \mathbf{\sigma} \, dv$$

From this we derive

$$\text{div} \mathbf{\sigma} + \rho b = \rho a$$

Using this relation in second relation we get

$$\int_{P_t} \mathbf{x} \wedge (\text{div} \mathbf{\sigma}) \, dv = \int_{\partial P_t} \mathbf{x} \wedge (\mathbf{\sigma}^T \mathbf{n}) \, da$$
Now from the right hand side, we get
\[
\int_{\partial P_t} \mathbf{x} \wedge (\sigma^T \mathbf{n}) \, da = \int_{\partial P_t} \epsilon_{ijk} x_j \sigma_{pk} n_p e_i \, da = \int_{P_t} \epsilon_{ijk} (x_j \sigma_{pk}) \, e_i \, dv = \int_{P_t} \epsilon_{ijk} (\delta_{jp} \sigma_{pk} + x_j \sigma_{pk,p}) \, e_i \, dv = \int_{P_t} \left[ \epsilon_{ijk} \sigma_{jk} e_i + \epsilon_{ijk} x_j \sigma_{pk,p} e_i \right] \, dv = \int_{P_t} \left[ \epsilon_{ijk} \sigma_{jk} e_i + \mathbf{x} \wedge (\text{div } \sigma) \right] \, dv
\]

Hence we get
\[
\int_{P_t} \epsilon_{ijk} \sigma_{jk} e_i = 0
\]

From this we get
\[
\epsilon_{ijk} \sigma_{jk} = 0
\]
Thus taking \( i = 1, 2, 3 \) in turn we obtained
\[
\sigma = \sigma^T
\]

### 9.2 Principal stresses and principal directions

Let \( \sigma(x, t) \) be the stress at a particular point. Let there exists direction \( \mathbf{n} \) such that stress vector \( \mathbf{t}(\mathbf{n}) = \sigma^T \mathbf{n} = \sigma \mathbf{n} \) is parallel to \( \mathbf{n} \). Thus
\[
\sigma \mathbf{n} = \sigma \mathbf{n}
\]

The values \( \sigma \) is called principal stresses and the corresponding directions \( \mathbf{n} \) are called the principal directions. Also the plane perpendicular to \( \mathbf{n} \) is called principal stress plane. Since \( \sigma \) is symmetric, there exists three mutually perpendicular directions and three principle stresses.

### 9.3 Cauchy-Stokes decomposition theorem

Let us consider the spatial distribution of velocity \( \mathbf{v} \) in two neighbouring points \( \mathbf{x}_0 \) and \( \mathbf{x} \). Using Taylor series (retaining only linear term and writing \( d\mathbf{x} = \mathbf{x} - \mathbf{x}_0 \)) we can write
\[
\mathbf{d} \mathbf{v} = b \mathbf{v}(\mathbf{x}, t) - \mathbf{x}(\mathbf{x}_0, t) = \mathbf{L} \, d\mathbf{x} = \mathbf{D} \, d\mathbf{x} + \mathbf{W} \, d\mathbf{x}
\]
Now let $ds$ be the length of line element between $x_0$ and $x$. Then
\[ ds^2 = dx \cdot dx = F dX \cdot F dX = dX \cdot F^T F X \]

Now
\[ \frac{D}{Dt} (ds^2) = dX \cdot \frac{D}{Dt} (F^T F) dX \]
\[ = dX \cdot (F^T F + F^T F^T) dX \quad (DP/Dt \text{ denoted by } \dot{P}) \]
\[ = dX \cdot (F^T L^T F + F^T L F) dX \]
\[ = F dX \cdot (L^T + L) F dX \]
\[ = 2dx \cdot Ddx \]

Thus we have
\[ \frac{1}{ds} \frac{D}{Dt} \frac{ds}{ds} = D_{ij} \frac{dx_i}{ds} \frac{dx_j}{ds} = D_{ij} n_i n_j \]

Since $D$ is real and symmetric and therefore has three real eigen values and three mutually orthogonal eigen vectors. Thus under the action of this term three infinitesimal fluid parcels resembling slender needles that are initially aligned with the eigen vectors will elongate or compress in their respective directions while remaining orthogonal to each other. Thus a spherical fluid parcels with its three axes aligned with the eigen vectors will deform (ellipsoidal), increasing or decreasing the the aspect ratios while maintaining its original directions. Thus $D$ represents deformation that preserves parcels orientation. To interpret vorticity, we note that the second term of can be written as
\[ \Omega \wedge dx \]

where $\Omega = \omega/2$. Thus under the action of this term the point particles rotate about the point $x_0$ with angular velocity that is equal to the half the vorticity of the fluid. Thus the vorticity vector is parallel to the angular velocity of the point particles and is equal to the twice the angular velocity of the point particles.

Thus the most general differential motion of a fluid element corresponds to a uniform translation, plus a rigid rotation plus a distortion.

*The following proof shows how a spherical fluid parcel deforms into a ellipsoid* Consider a small sphere of radius $dr$ at time $t$. Let the axis be the principal axes of $D$. Let the particles on the sphere of center $x$ is $x + n \, dr$ where $n$ is a unit vector. The material coordinate of the center and the point on the sphere are $X$ and $X + dX$. Thus
\[ n_i dr = \left( \frac{\partial x_i}{\partial X_j} \right) \dot{X}_j dX_j \]
In the interval from $t$ to $t + dt$ the center moves from $\mathbf{x}(\mathbf{X}, t)$ to $\mathbf{x}(\mathbf{X}, t + dt)$. If $\mathbf{dy}$ is the relative position of the of a particle on the surface relative to the center, then we have

$$
dy_i = x_i(\mathbf{X} + d\mathbf{X}, t + dt) - x_i(\mathbf{X}, t + dt) = \left( \frac{\partial x_i}{\partial X_j} \right)_{t + dt} dX_j
$$

Now

$$
\left( \frac{\partial x_i}{\partial X_j} \right)_{t + dt} = \left( \frac{\partial x_i}{\partial X_j} \right)_t + dt \frac{D}{Dt} \left( \frac{\partial x_i}{\partial X_j} \right)_t
$$

We can also write the first relation (using inverse mapping) as

$$
dX_j = \left( \frac{\partial X_j}{\partial x_k} \right)_t n_k dr
$$

Thus we get

$$
dy_i = n_i dr + \left( \frac{\partial v_i}{\partial X_j} \right) dX_j dt$$

$$
= n_i dr + \left( \frac{\partial v_i}{\partial X_j} \right) \left( \frac{\partial X_j}{\partial x_k} \right)_t n_k dr dt$$

$$
= n_i dr + \left( \frac{\partial v_i}{\partial x_k} \right) n_k dr dt = A_{ik} n_k dr\ dt$$

where due to principal axes of $\mathbf{D}$

$$
A_{ii} = 1 + D_{ii} dt, \quad i = k \quad \text{no sum}
$$

and

$$
A_{ik} = \frac{1}{2} (v_{i,k} - v_{k,i}), \quad i \neq k
$$

The off diagonal term represent rigid rotation. In the absence of rotation we have since $n_i$ are unit vector,

$$
1 = \frac{dy_i dy_i}{(1 + D_{ii} dt)^2 dr^2}
$$

and this is an infinitesimal ellipsoid whose axes are coincident with the principal axes of stretching of length $(1 + D_{ii} dt) dr, i = 1, 2, 3$ (no sum). Thus in the complete deformation, a small sphere is distorted into an ellipsoid and rotated.
10 Constitutive equations

10.1 Stokesian Fluid

A Stokesian fluid satisfies the following assumption

I. The stress tensor is a continuous function of the rate of strain tensor $D_{ij}$ and local thermodynamic state but independent of other kinematical quantities.

II. The fluid is homogeneous i.e. $\sigma_{ij}$ does not depend explicitly on $x$

III. The fluid is isotropic, i.e. there is no preferred direction.

IV. When there is no deformation ($D_{ij} = 0$) the stress is hydrostatic i.e. $\sigma_{ij} = -p\delta_{ij}$

The first assumption implies that relation between stress and strain is independent of the rigid body rotation by $W_{ij}$. The thermodynamic variables, for example, pressure and temperature, will be carried along throughout the discussion without specific mention. Due to homogeneous nature the stress tensor depend on position through the variation of $D_{ij}$. The third assumption is isotropy and we first show that it implies that the principal direction of two tensors coincide. To express isotropy as an equation we write $\sigma_{ij} = f_{ij}(D_{ij})$, then if there is not preferred direction, we also have $\bar{\sigma}_{ij} = f_{ij}(\bar{D}_{ij})$. Let us write

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

where $\tau_{ij}$ is also isotropic and vanish when there is no deformation.

Since $\tau$ is isotropic we must have

$$\tau(QDQ^T) = Q\tau(D)Q^T$$

Choose the coordinate axes as the principal axes of $D$. Then in this coordinate we have

$$[D_{ij}] = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}$$

and

$$\tau_{ij} = \tau_{ij}(D_1, D_2, D_3)$$
Choose $Q$ as

$$[Q_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Then $\bar{D}_{ij} = D_{ij}$ and

$$[\bar{\tau}_{ij}] = \begin{bmatrix} \tau_{11} & -\tau_{12} & -\tau_{13} \\ -\tau_{12} & \tau_{22} & \tau_{23} \\ -\tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix}$$

Using $\bar{\tau}_{ij} = \tau_{ij}$ we get $\sigma_{12} = \sigma_{13} = 0$. Similarly we can show that $\sigma_{23} = 0$. Thus both the $D$ and $\tau$ are diagonal matrix and hence same principal axes. Thus we have

$$\tau_{11} = \tau_1 = F(D_1, D_2, D_3), \tau_{22} = \tau_2 = F_2(D_1, D_2, D_3), \tau_{33} = \tau_3 = F_3(D_1, D_2, D_3)$$

Next choose $Q$ as

$$[Q_{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Now we have

$$[\bar{D}_{ij}] = \begin{bmatrix} D_2 & 0 & 0 \\ 0 & D_3 & 0 \\ 0 & 0 & D_1 \end{bmatrix}, \quad [\bar{\tau}_{ij}] = \begin{bmatrix} \tau_2 & 0 & 0 \\ 0 & \tau_3 & 0 \\ 0 & 0 & \tau_1 \end{bmatrix}$$

Thus we have

$$F(D_2, D_3, D_1) = F_2(D_1, D_2, D_3)$$
$$F_2(D_2, D_3, D_1) = F_3(D_1, D_2, D_3)$$
$$F_3(D_2, D_3, D_1) = F(D_1, D_2, D_3)$$

Hence $\tau_1, \tau_2, \tau_3$ can be expressed as a single function $F(D_1, D_2, D_3)$ as

$$\tau_1 = F(D_1, D_2, D_3), \tau_2 = F(D_2, D_3, D_1), \tau_3 = F(D_3, D_1, D_2)$$

Finally choose $Q$ as

$$[Q_{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Then we have

\[
\tilde{D}_{ij} = \begin{bmatrix}
D_2 & 0 & 0 \\
0 & D_1 & 0 \\
0 & 0 & D_3 \\
\end{bmatrix} \quad \tilde{\tau}_{ij} = \begin{bmatrix}
\tau_2 & 0 & 0 \\
0 & \tau_1 & 0 \\
0 & 0 & \tau_3 \\
\end{bmatrix}
\]

Thus we have

\[
F(D_2, D_1, D_3) = F(D_2, D_3, D_1), \quad F(D_1, D_3, D_2) = F(D_1, D_2, D_3)
\]

Now the equations

\[
\alpha + \beta D_1 + \gamma D_1^2 = F(D_1, D_2, D_3) \\
\alpha + \beta D_2 + \gamma D_2^2 = F(D_2, D_3, D_1) \\
\alpha + \beta D_3 + \gamma D_3^2 = F(D_3, D_1, D_2)
\]

give solutions of \(\alpha, \beta, \gamma\) as functions of \(D_1, D_2, D_3\). Also since \(F(D_1, D_2, D_3)\) has symmetry, the above solutions are unaltered if any pair of \(D_1, D_2, D_3\) are interchanged. Hence \(\alpha, \beta, \gamma\) can be expressed as functions of

\[
D_1 + D_2 + D_3 = I_1 \\
D_1 D_2 + D_2 D_3 + D_3 D_1 = I_2 \\
D_1 D_2 D_3 = I_3
\]

Also we have

\[
\begin{bmatrix}
\tau_1 & 0 & 0 \\
0 & \tau_2 & 0 \\
0 & 0 & \tau_3 \\
\end{bmatrix} = \alpha \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} + \beta \begin{bmatrix}
D_1 & 0 & 0 \\
0 & D_2 & 0 \\
0 & 0 & D_3 \\
\end{bmatrix} + \gamma \begin{bmatrix}
D_1^2 & 0 & 0 \\
0 & D_2^2 & 0 \\
0 & 0 & D_3^2 \\
\end{bmatrix}
\]

Multiplying from left with \(Q\) and from right with \(Q^T\) we get

\[
Q \begin{bmatrix}
\tau_1 & 0 & 0 \\
0 & \tau_2 & 0 \\
0 & 0 & \tau_3 \\
\end{bmatrix} Q^T = \alpha \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} + \beta Q \begin{bmatrix}
D_1 & 0 & 0 \\
0 & D_2 & 0 \\
0 & 0 & D_3 \\
\end{bmatrix} Q^T + \gamma Q \begin{bmatrix}
D_1 & 0 & 0 \\
0 & D_2 & 0 \\
0 & 0 & D_3 \\
\end{bmatrix} Q^T
\]
which gives

$$\tau_{ij} = \alpha \delta_{ij} + \beta D_{ij} + \gamma D_{ik} D_{kj}$$

Thus

$$\sigma_{ij} = (-p + \alpha) \delta_{ij} + \beta D_{ij} + \gamma D_{ik} D_{kj}$$

Here $p$ depends on the thermodynamic state but $\alpha, \beta, \gamma$ depend as well on the invariants of the tensor. If the fluid is compressible, the thermodynamic pressure is well defined and we take $p$ equal to this. Then by forth assumption we must have $\alpha = 0$. If the fluid is incompressible, the thermodynamic pressure is not defined and pressure has to be taken as one of the fundamental dynamic variables. In this case we absorb $\alpha$ into pressure $p$ and write

$$\sigma_{ij} = -p \delta_{ij} + \beta D_{ij} + \gamma D_{ik} D_{kj}$$

which ensures that $\sigma$ reduces to hydrostatic form when the deformation vanishes.

### 10.2 Newtonian Fluid

A Newtonian fluid is linear Stokesian fluid i.e. the stress component depends linearly on the rates of strain tensor. Writing

$$\sigma_{ij} = -p \delta_{ij} + \tau_{ij}$$

the Newtonian fluid is given by the following relations (express in terms of principal axes)

$$\begin{align*}
\tau_1 &= a_{11} D_1 + a_{12} D_2 + a_{13} D_3 \\
\tau_2 &= a_{21} D_1 + a_{22} D_2 + a_{23} D_3 \\
\tau_3 &= a_{31} D_1 + a_{32} D_2 + a_{33} D_3
\end{align*}$$

Due to isotropy, any permutation of $D$’s will affect the same permutation of $\tau$’s. Now permute $D_1, D_2, D_3$ to $D_3, D_1, D_2$ and rearrange to obtain

$$\begin{align*}
\tau_3 &= a_{12} D_1 + a_{13} D_3 + a_{11} D_3 \\
\tau_1 &= a_{22} D_1 + a_{23} D_2 + a_{21} D_3 \\
\tau_2 &= a_{32} D_1 + a_{33} D_2 + a_{31} D_3
\end{align*}$$

Comparing these two sets of equations; for example

$$\tau_1 = a_{11} D_1 + a_{12} D_2 + a_{13} D_3 = a_{22} D_1 + a_{23} D_2 + a_{21} D_3$$
gives $a_{11} = a_{22}, a_{12} = a_{23}, a_{13} = a_{21}$

Doing this for all and for the sets of equation obtain by permuting $D_1, D_2, D_3$ to $D_2, D_3, D_1$ we find

$$a_{11} = a_{22} = a_{33} = \lambda + 2\mu$$
$$a_{12} = a_{21} = a_{23} = a_{32} = a_{13} = a_{31} = \lambda$$

Thus

$$\tau_i = \lambda(D_1 + D_2 + D_3) + 2\mu D_i = \lambda tr(D) + 2\mu D_i$$

Transforming to a general coordinate system we get

$$\tau_{ij} = \lambda tr(D)\delta_{ij} + 2\mu D_{ij}$$

Hence

$$\sigma_{ij} = -p\delta_{ij} + \lambda(\nabla \cdot v)\delta_{ij} + 2\mu D_{ij}$$

11 Fluids

A fundamental characteristic of any fluid is that the action of shear stresses, no matter how small they might be, will cause the fluid to deform continuously as long as the shear stresses act. Thus a fluid at rest the stress vector on any plane is normal to that plane. Thus every plane is a principal plane and consequently every direction is an eigen vector of the stress tensor. If $n_1$ and $n_2$ are two arbitrary directions and $\sigma_1$ and $\sigma_2$ are two eigen values then

$$\boldsymbol{\sigma} n_1 = \sigma_1 n_1 \quad \text{and} \quad \boldsymbol{\sigma} n_2 = \sigma_2 n_2$$

Since

$$n_1 \cdot \boldsymbol{\sigma} n_2 = n_2 \cdot \boldsymbol{\sigma} n_1$$

we get

$$(\sigma_1 - \sigma_2) n_1 \cdot n_2 = 0$$

Since $n_1$ and $n_2$ arbitrary, we have $\sigma_1 = \sigma_2 = -p$. Hence we write

$$\boldsymbol{\sigma} = -p I \quad \text{or} \quad \sigma_{ij} = -p\delta_{ij}$$

The scalar $p$ is the magnitude of the compressive normal stress and is known as hydrostatic pressure.
12 Compressible and Incompressible Fluids

We define an incompressible fluid to be the one for which the density of every particle remains the same at all times regardless of the state of stress. Thus for an incompressible fluid

\[ \frac{D\rho}{Dt} = 0 \]

It then follows that

\[ \text{div } \mathbf{v} = 0 \]

An incompressible fluid need not have a spatially uniform density. If the density is also uniform, it is referred to as homogeneous fluid for which \( \rho \) is constant everywhere.

The compressible fluids are those for which the density change appreciably with pressure.

13 Equations of hydrostatics

The term hydrostatic refers to the study of fluid at rest i.e. \( \mathbf{v} = 0 \). Thus with \( \sigma_{ij} = -p\delta_{ij} \) the equilibrium equations become

\[ \frac{\partial p}{\partial x_i} = \rho b_i \quad \text{or} \quad \text{grad } p = \rho \mathbf{b} \]

In case the only force acting on the body is the force of gravity, then with \( x_3 \)-axis pointing upwards, we get

\[ \frac{\partial p}{\partial x_1} = 0 \\
\frac{\partial p}{\partial x_2} = 0 \\
\frac{\partial p}{\partial x_3} = -\rho g \]

This shows that the pressure \( p \) is independent of \( x_1 \) and \( x_2 \). In case of homogeneous fluid i.e. (constant density) we further get

\[ p = -\rho gx_3 + p_0 \]

where \( p_0 \) is a constant.
14 Newtonian fluid

Since the state of stress for a fluid under rigid body motion (including rest) is given by isotropic, it is natural to decompose the stress tensor into two parts

$$\sigma_{ij} = -p \delta_{ij} + \tau_{ij}$$

where the values of $\tau_{ij}$ depends on the rate and/or higher rates of deformation such that they are zero when the fluid is under rigid body motion (i.e. zero rates of deformation) and $p$ is a scalar whose value does not depend explicitly on these rates.

We idealized the fluid as a Newtonian fluid if

a. The values of $\tau_{ij}$ at any time $t$ depend linearly on the components of rate of deformation tensor

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$$

at that time and not on any other kinematic quantities.

b. The fluid is isotropic

Following the same argument as the linear isotropic solid we get

$$\tau_{ij} = \lambda D_{k,k} \delta_{ij} + 2\mu D_{ij}$$

where $\lambda$ and $\mu$ are called viscosity coefficient.

For a fluid under rigid body motion (i.e. zero rate of deformation) we have

$$p = -\frac{1}{3} \sigma_{ii},$$

i.e. pressure is the total compressive normal stress on any plane as well as the mean of the normal stresses. But in this case we have

$$-p + (\lambda + 2\mu/3)D_{kk} = -\frac{1}{3} \sigma_{ii}$$

where $k = \lambda + 2\mu/3$ is known as coefficient of bulk viscosity. It is clear that when $D_{ij}$ are non zero, $p$ is neither the total compressive normal stress on any plane unless the viscous components happen to be zero as well as is not the mean of the normal stresses. Thus we can interpret pressure such that $-p \delta_{ij}$ is that part of $\sigma_{ij}$ which does not depend explicitly on the rate of deformation.
If we enforce the condition that the pressure is mean of the compressive normal stresses, we must have

$$\lambda = -2\mu/3$$

which is called Stoke’s condition. Thus choosing $\mu$ as the only scaler constant we write

$$\sigma_{ij} = -p\delta_{ij} + 2\mu \left(D_{ij} - \frac{1}{3} D_{kk}\right)$$

15 Law of conservation of energy

The law states that the material derivative of kinetic plus internal energies is equal to the sum of the rate of work of the surface and body forces, plus all other energies that enter and leave the body per unit time. Other energies may include thermal, electrical, magnetical etc. Here we only considered mechanical and thermal energies. Let $e$ be the specific internal energy or internal energy per unit mass. The conservation of energy is stated as

$$\frac{D}{Dt} \int_V \rho \left(e + \frac{1}{2} v_i v_i \right) dv = \int_V \rho b_i v_i \ dv + \int_S t_i^{(n)} v_i \ dS + \int_V \rho r \ dv - \int_S q_i n_i \ dS$$

where $r$ is specific rate at which heat is produced by internal sources and $q$ is the heat flux vector. Now the term on the left hand side can be written as

$$\frac{D}{Dt} \int_V \rho \left(e + \frac{1}{2} v_i v_i \right) dv = \int_V \rho \frac{De}{Dt} \ dv + \int_V \frac{D}{Dt} (\rho v_i) \ dv$$

$$= \int_V \rho \frac{De}{Dt} \ dv + \int_V \rho (\rho b_i + \sigma_{ij,j}) \ dv$$

$$= \int_V \rho \frac{De}{Dt} \ dv + \rho v_i b_i + (\sigma_{ij} v_i)_j - \sigma_{ij} v_{ij} \ dv + \int_S t_i^{(n)} v_i \ dS$$

Hence we have

$$\int_V \rho \frac{De}{Dt} \ dv = \int_V \sigma_{ij} D_{ij} \ dv + \int_V \rho r \ dv - \int_S \nabla \cdot q \ dv$$

Since the volume $V$ is arbitrary we get

$$\rho \frac{De}{Dt} = \sigma_{ij} D_{ij} + \rho r - \nabla \cdot q$$
The term $\sigma_{ij}D_{ij}$ is called stress power. For Newtonian Viscous fluid we can explicitly calculate the stress power. We have

$$\sigma_{ij}D_{ij} = (-p\delta_{ij} + \tau_{ij})D_{ij} = -p\nabla \cdot \mathbf{v} + \Phi$$

where $\Phi$ is the dissipation function. The function $\Phi$ is positive definite since we have

$$\Phi = 2\mu \left( D_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{v})\delta_{ij} \right) D_{ij} = 2\mu \left( D_{ij}D_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{v})^2 \right) = 2\mu \left( D_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{v})\delta_{ij} \right)^2 > 0$$

Alternative form of energy equation is also possible using definition of enthalpy

$$h = e + \frac{p}{\rho}$$

We have

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho} \frac{Dp}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt}$$

Thus we have

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \Phi + \nabla \cdot \mathbf{q}$$

We may relate the heat flux to temperature by Fourier’s law as

$$q_i = -\kappa T_i$$

Then we get

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \Phi + (\kappa T_i)_i$$

To close the system we use (i) a thermodynamic equation like $e = e(T, p)$, the simplest being $e = c_v T$ or $h = c_p T$ and (ii) equation of state $p = \rho RT$

16 Navier Stokes Equation

These equations describe the conservation of mass, momentum and energy. These are

$$\frac{D\rho}{Dt} + \rho v_{k,k} = 0$$

$$\rho \frac{Dv_i}{Dt} = -p_i + \tau_{ij,j} + \rho b_i$$

$$\rho \frac{De}{Dt} = -\rho v_{k,k} + \Phi + (\kappa T_i)_i$$
where
\[ \tau_{ij} = \mu \left( v_{i,j} + v_{j,i} - \frac{1}{3} v_{k,k} \delta_{ij} \right) \]
\[ \Phi = \tau_{ij} D_{ij} = \tau_{ij} v_{i,j} \]

These are supplemented with the thermodynamic relation and equation of state of a gas
\[ e = e(T, p) \]
\[ p = \rho RT \]

These gives seven equations in seven unknowns \( \rho, p, T, v_i, e \).

17 Incompressible Navier Stokes Equation

For incompressible flow we shall always treat flow with density constant (technically not most general!). For an incompressible fluid \( D_{kk} = 0 \). Thus we have
\[ \sigma_{ij} = -p \delta_{ij} + 2\mu D_{ij} \]
and we now have \( p = -\sigma_{ii}/3 \) which has the meaning of mean normal compressive stress. Substituting into the equations of motion
\[ \rho \left( \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \rho b_i + \frac{\partial \sigma_{ij}}{\partial x_j} \]
which upon simplification gives
\[ \rho \left( \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \rho b_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \]

In componentwise we write the above as
\[
\begin{align*}
\rho \left( \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} \right) &= \rho b_1 - \frac{\partial p}{\partial x_1} + \mu \left( \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} \right) v_1 \\
\rho \left( \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} \right) &= \rho b_2 - \frac{\partial p}{\partial x_2} + \mu \left( \frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2} \right) v_2 \\
\rho \left( \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3} \right) &= \rho b_3 - \frac{\partial p}{\partial x_3} + \mu \left( \frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2} \right) v_3
\end{align*}
\]

There are four unknowns \( v_1, v_2, v_3 \) and \( p \) in the three equations. The fourth equation is supplied by the condition on incompressibility i.e. mass conservation \( \text{div } \mathbf{v} = 0 \) i.e.
\[ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \]
These four equations are called Navier Stokes equations of motion for a incompressible fluid. Since the energy equations decouple from the mass conservation and momentum equations and energy equation need to be solved only when we need temperature. For energy equation we have
\[ \rho \frac{D\varepsilon}{Dt} = -pv_{k,k} + \Phi + (\kappa T)_i. \]
Almost always safe to assume that \( \Phi \ll 1 \). And since the fluid is incompressible we have (using \( e = c_v T \))
\[ \rho c_v \frac{DT}{Dt} = (\kappa T)_i. \]
The Navier-Stokes equation in invariant form can be written as
\[
\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\text{grad} \mathbf{v}) \mathbf{v} \right] = \rho \mathbf{b} - \text{grad} \, p + \mu \text{div} \, (\text{grad} \, \mathbf{v})^T \\
\text{div} \, \mathbf{v} = 0
\]

18 Cylindrical coordinate

To derive equations in cylindrical coordinate we need to have the expressions for \( \text{grad} f \), \( \text{grad} \mathbf{v} \) and \( \text{div} \mathbf{A} \) for scalar, vector and tensor fields \( f, \mathbf{v} \) and \( \mathbf{A} \) respectively. Let \((r, \phi, z)\) is the cylindrical coordinate with unit base vectors \( \mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z \). These can be expressed in terms of cartesian vectors \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) as
\[
\mathbf{e}_r = \cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_2 \\
\mathbf{e}_\phi = -\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2 \\
\mathbf{e}_z = \mathbf{e}_3
\]
From these we get \( d\mathbf{e}_r = d\phi \mathbf{e}_\phi \) and \( d\mathbf{e}_\phi = -d\phi \mathbf{e}_r \). From the position vector \( \mathbf{r} = r \mathbf{e}_r + z \mathbf{e}_z \) we get
\[ d\mathbf{r} = dre_r + rd\phi \mathbf{e}_\phi + dz \mathbf{e}_z \]

18.1 Components of \( \text{grad} f \)

From definition we have
\[
df = (\text{grad} f) \cdot d\mathbf{r} = [(\text{grad} f)_r \mathbf{e}_r + (\text{grad} f)_\phi \mathbf{e}_\phi + (\text{grad} f)_z \mathbf{e}_z] \cdot [dre_r + rd\phi \mathbf{e}_\phi + dz \mathbf{e}_z] \\
= (\text{grad} f)_r dr + (\text{grad} f)_\phi r d\phi + (\text{grad} f)_z dz
\]
From calculus
\[ df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \phi} d\phi + \frac{\partial f}{\partial z} dz \]

Comparing we get
\[ \text{grad} f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \phi} e_\phi + \frac{\partial f}{\partial z} e_z \]

### 18.2 Components of grad v

We have
\[ v(r, \phi, z) = v_r(r, \phi, z)e_r + v_\phi(r, \phi, z)e_\phi + v_z(r, \phi, z)e_z \]

From the definition we have
\[ dv = (\text{grad} v)dr = (\text{grad} v)(dr e_r + r d\phi e_\phi + dz e_z) \]
\[ = dr(\text{grad} v)e_r + r d\phi(\text{grad} v)e_\phi + dz(\text{grad} v)e_z \]

Now
\[ (\text{grad} v)e_r = (\text{grad} v)_{rr} e_r + (\text{grad} v)_{r\phi} e_\phi + (\text{grad} v)_{rz} e_z \]
\[ (\text{grad} v)e_\phi = (\text{grad} v)_{r\phi} e_r + (\text{grad} v)_{\phi\phi} e_\phi + (\text{grad} v)_{z\phi} e_z \]
\[ (\text{grad} v)e_z = (\text{grad} v)_{rz} e_r + (\text{grad} v)_{\phi z} e_\phi + (\text{grad} v)_{zz} e_z \]

Thus
\[ dv = [(\text{grad} v)_{rr} e_r + (\text{grad} v)_{r\phi} e_\phi + (\text{grad} v)_{rz} e_z]dr \]
\[ + [(\text{grad} v)_{r\phi} e_r + (\text{grad} v)_{\phi\phi} e_\phi + (\text{grad} v)_{z\phi} e_z]r d\phi \]
\[ + [(\text{grad} v)_{rz} e_r + (\text{grad} v)_{\phi z} e_\phi + (\text{grad} v)_{zz} e_z]dz \]

Now
\[ dv = dv_re_r + v_r de_r + dv_\phi e_\phi + v_\phi de_\phi + dv_z e_z \]

Also from calculus
\[ dv_r = \frac{\partial v_r}{\partial r} dr + \frac{\partial v_r}{\partial \phi} d\phi + \frac{\partial v_r}{\partial z} dz \]
\[ dv_\phi = \frac{\partial v_\phi}{\partial r} dr + \frac{\partial v_\phi}{\partial \phi} d\phi + \frac{\partial v_\phi}{\partial z} dz \]
\[ dv_z = \frac{\partial v_z}{\partial r} dr + \frac{\partial v_z}{\partial \phi} d\phi + \frac{\partial v_z}{\partial z} dz \]
Using these we get

\[ dv = \left[ \frac{\partial v_r}{\partial r} dr + \left( \frac{\partial v_r}{\partial \phi} - v_\phi \right) d\phi + \frac{\partial v_r}{\partial z} dz \right] e_r + \left[ \frac{\partial v_\phi}{\partial r} dr + \left( \frac{\partial v_\phi}{\partial \phi} + v_r \right) d\phi + \frac{\partial v_\phi}{\partial z} dz \right] e_\phi + \left[ \frac{\partial v_z}{\partial r} dr + \frac{\partial v_z}{\partial \phi} d\phi + \frac{\partial v_z}{\partial z} dz \right] e_z \]

Comparing the above two form we get grad \( v \) in matrix form as

\[
[\text{grad} v] = \begin{bmatrix}
\frac{\partial v_r}{\partial r} & \frac{1}{r} \left( \frac{\partial v_r}{\partial \phi} - v_\phi \right) & \frac{\partial v_r}{\partial z} \\
\frac{\partial v_\phi}{\partial r} & \frac{1}{r} \left( \frac{\partial v_\phi}{\partial \phi} + v_r \right) & \frac{\partial v_\phi}{\partial z} \\
\frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \phi} & \frac{\partial v_z}{\partial z}
\end{bmatrix}
\]

### 18.3 \( \text{div} v \)

Using definiton

\[ \text{div} v = \text{tr}(\text{grad} v) = (\text{grad} v)_{rr} + (\text{grad} v)_{\phi\phi} + (\text{grad} v)_{zz} \]

\[ = \frac{\partial v_r}{\partial r} + \frac{1}{r} \left( \frac{\partial v_\phi}{\partial \phi} + v_r \right) + \frac{\partial v_z}{\partial z} \]

### 18.4 Curl \( v \)

The antisymmetric part of \( \text{grad} v \) in matrix form is

\[
[\text{grad} v]^A = \begin{bmatrix}
0 & \frac{1}{2} \left( \frac{\partial v_r}{\partial \phi} - v_\phi \right) & \frac{1}{2} \left( \frac{\partial v_r}{\partial z} - \frac{\partial v_\phi}{\partial r} \right) \\
-\frac{1}{2} \left( \frac{\partial v_r}{\partial \phi} - v_\phi \right) & 0 & \frac{1}{2} \left( \frac{\partial v_\phi}{\partial z} - \frac{1}{r} \frac{\partial v_r}{\partial \phi} \right) \\
-\frac{1}{2} \left( \frac{\partial v_\phi}{\partial z} - \frac{\partial v_r}{\partial r} \right) & \frac{1}{2} \left( \frac{\partial v_\phi}{\partial z} - \frac{1}{r} \frac{\partial v_\phi}{\partial r} \right) & 0
\end{bmatrix}
\]

Since curl is twice the dual vector of antisymmetric part of \( \text{grad} v \), we have

\[ \text{Curl} v = \left( \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) e_r + \left( \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) e_\phi + \left( \frac{v_\phi}{r} + \frac{\partial v_\phi}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \phi} \right) e_z \]

### 18.5 Components of div \( A \)

We use the identity

\[ \text{div}(A v) = (\text{div} A) \cdot v + \text{tr}(\text{grad}(v) A) \]
To get component along $e_r$ we take $v = e_r$. Thus we have

$$(\text{div} \mathbf{A})_r = (\text{div} \mathbf{A}) \cdot e_r = \text{div}(A e_r) - \text{tr}(\text{grad}(e_r) \mathbf{A})$$

$$= \frac{\partial A_{rr}}{\partial r} + \frac{1}{r} \frac{\partial A_{r\phi}}{\partial \phi} + \frac{\partial A_{rz}}{\partial z} + \frac{A_{r\phi}}{r} - \text{tr}(\text{grad}(e_r) \mathbf{A})$$

$$= \frac{\partial A_{rr}}{\partial r} + \frac{1}{r} \frac{\partial A_{r\phi}}{\partial \phi} + \frac{\partial A_{rz}}{\partial z} + \frac{A_{rr} - A_{\phi\phi}}{r}$$

Similarly

$$(\text{div} \mathbf{A})_\phi = \frac{\partial A_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial A_{\phi\phi}}{\partial \phi} + \frac{\partial A_{rz}}{\partial z} + \frac{A_{r\phi} + A_{r\phi}}{r}$$

and

$$(\text{div} \mathbf{A})_z = \frac{\partial A_{rz}}{\partial r} + \frac{1}{r} \frac{\partial A_{\phi z}}{\partial \phi} + \frac{\partial A_{zz}}{\partial z} + \frac{A_{rz}}{r}$$

19 Navier Stokes equations in cylindrical coordinates

The continuity equation takes the form

$$\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0$$

and the linear momentum equations are given by

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \phi} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + b_r + \frac{\mu}{\rho} \left[ \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \phi^2} \right]$$

$$+ \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi}$$

$$\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \phi} + v_z \frac{\partial v_\phi}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial \phi} + b_\phi + \frac{\mu}{\rho} \left[ \frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial \phi^2} \right]$$

$$+ \frac{\partial^2 v_\phi}{\partial z^2} + \frac{1}{r} \frac{\partial v_\phi}{\partial r}$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_z}{\partial \phi} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + b_z + \frac{\mu}{\rho} \left[ \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right]$$
20 Navier Stokes equations in spherical coordinates

The continuity equation takes the form
\[ \frac{\partial v_r}{\partial r} + \frac{2v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} = 0 \]

and the linear momentum equations are given by
\[
\begin{align*}
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{\partial \theta} + v_\phi \frac{\partial v_r}{\partial \phi} & - \frac{v_\theta^2}{r} - \frac{v_\phi^2}{r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + b_r + \frac{\mu}{\rho} \left[ \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} \right] \\
\frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2v_r}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_\theta}{r^2} - \frac{2v_\phi \cot \theta}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} & = \frac{1}{\rho} \frac{\partial p}{\partial r} + b_\theta + \frac{\mu}{\rho} \left[ \frac{\partial^2 v_\theta}{\partial r^2} \right] \\
\frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2v_r}{r} \frac{\partial v_\phi}{\partial \theta} - \frac{v_\phi}{r^2} - \frac{2 \cot \theta \frac{\partial v_\phi}{\partial \phi}}{r^2 \sin \theta} & = - \frac{1}{\rho r \sin \theta} \frac{\partial p}{\partial \phi} + b_\phi + \frac{\mu}{\rho} \left[ \frac{\partial^2 v_\phi}{\partial r^2} \right]
\end{align*}
\]

21 Exact Solutions

21.1 Couette Flow

For steady flow in the absence of body forces the Navier-Stokes equation reduce to

\[ \rho (\nabla v) v = \mu \nabla \cdot ((\nabla v)^T) - \nabla p \]

\[ \nabla \cdot v = 0 \]

Consider the plane velocity field
\[ v(x) = v_1(x_1, x_2) e_1 \]

Now the mass conservation implies that
\[ \frac{\partial v_1}{\partial x_1} = 0 \]
so that $v_1 = v_1(x_2)$. Further the matrix of $\nabla v$ is

$$
[\nabla v] = \begin{bmatrix}
0 & \frac{\partial v_1}{\partial x_2} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

Thus $\nabla \cdot v = 0$ and $(\nabla v)v = 0$. Thus the equation of motion reduces to

$$
\mu \frac{\partial^2 v_1}{\partial x_2^2} = \frac{\partial p}{\partial x_1}, \quad \frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3} = 0
$$

This implies that $p = p(x_1)$. Also

$$
\frac{\partial}{\partial x_1} \left( \frac{\partial p}{\partial x_1} \right) = \mu \frac{\partial^2}{\partial x_2^2} \frac{\partial v_1}{\partial x_1} = 0
$$

This implies that $\partial p/\partial x_1$ is a constant. We now consider two specific cases consistent with the above flow.

**Problem 1 (Plane Couette Flow).** Consider the flow between two infinite flat plates, one at $x_2 = 0$ and one at $x_2 = h$. The bottom plate is stationary and the top plate is moving in the $x_1$ direction with velocity $v$. Thus the boundary conditions are

$$
v_1(0) = 0, \quad v_1(h) = v
$$

Let us assume that $\partial p/\partial x_1 = 0$ i.e. $p = \text{constant}$. Thus we have

$$
v_1 = \alpha + \beta x_2
$$

Using the boundary condition we get

$$
v_1 = v x_2 / h
$$

The stress components are

$$
\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \quad \sigma_{12} = \sigma_{21} = \mu v / h
$$

Thus the force per unit area exerted by the fluid on the top plate has a normal component $p$ and a tangential component $\mu v / h$. This fact furnishes a method of determining the viscosity of a fluid.

**Problem 1 (Plane Poiseuille Flow).** Now the two plates are fixed in space. Then the relevant boundary conditions are

$$
v_1(0) = v_1(h) = 0
$$

If the pressure is constant then velocity field would be identically zero. We therefore allow pressure drop. In particular let

$$
\frac{\partial p}{\partial x_1} = -\delta
$$
with $\delta$ (the pressure drop per unit length) constant. Now the solution is given by

$$v_1 = -\frac{\delta x_2^2}{2\mu} + \alpha + \beta x_2$$

which using the boundary condition gives

$$v_1 = \frac{\delta x_2}{2\mu} (h - x_2)$$

and the velocity distribution is parabolic. Also the stress components are given by

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \quad \sigma_{12} = \sigma_{21} = \delta\left(\frac{h}{2} - x_2\right)$$

Thus the shearing stresses are maximum at the wall and vanishes at the centre. The volume discharge $Q$ through a cross section is

$$Q = \int_0^h v_1 dx_2 = \frac{h^3}{12}\delta/\mu$$

Since the discharge can be easily measured as well as the pressure drop, this formula yields a convenient method of determining the viscosity.

**Problem 1 (General Couette Flow).** In this case either of the two surfaces is moving at constant speed and there is also external pressure gradient. The solution can be obtained by superimposing the two solutions obtained earlier since the governing equations are linear. Thus we get

$$v_1 = vx_2/h + \frac{\delta x_2^2}{2\mu} (h - x_2)$$

If $\delta > 0$ then the pressure gradient will assist the motion induced by the upper surface. On the other hand it will resist the induced motion if $\delta < 0$. In this case we may have reverse flow near the lower surface.

### 21.2 Poiseuille Flow

The steady flow of viscous fluid in a pipe of arbitrary but constant cross section is referred to as Poiseuille flow. Let the cross section is in the $xy$ plane and the steady flow is in the $z$ direction.

Since the flow is unidirectional, we have $v = (0, 0, w)$. Now equation of mass conservation gives

$$\frac{\partial w}{\partial z} = 0 \implies w = w(x, y)$$

Thus $(\nabla v)w = 0$. The equation of motion reduces to

$$-\nabla p + \mu \nabla^2 v = 0$$
Writing components wise we get \( p_x = p_y = 0 \). Thus \( p = p(z) \). Thus equation for \( w \) becomes
\[
\left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{1}{\mu} \frac{dp}{dz}
\]
Now taking the right hand side can atmost be function of \( z \). Taking derivative with respect to \( z \) results in \( dp/dz \) equals to a constant, \(-G\) say. Thus we have
\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{G}{\mu}
\]
For very special geometries, closed for solution can be found. For other, no closed form solution exist. In those cases we might sought solution using series form.

Problem 1 (Poiseuille Flow: Circular cross section). We now use polar coordinate in the xy-plane. This effectively means we are using cylindrical coordinate for the original problem. Let \( a \) be the radius of the cylinder. Then transformaing to \( x = r \cos \theta \) and \( y = r \sin \theta \) we get
\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) = -\frac{G}{\mu}
\]
Integrating this we get
\[
w = -\frac{G \cdot r^2}{4 \mu} + A \log r + B
\]
The boundary condition is that \( w(0) \) is finite and \( w(a) = 0 \). Thus we have
\[
w = \frac{G}{4\mu} (a^2 - r^2)
\]
The maximum velocity occurs at $r = 0$ and
\[ w_{\text{max}} = \frac{G}{4\mu} \]

The same solution could be obtained if we had guessed the solution in the form
\[ w(x, y) = \alpha(x^2 + y^2 - a^2) \]

Note that this satisfies the boundary condition and the constant $\alpha$ is obtained by substituting this in the momentum equation for $w$.

The flux through the pipe is given by
\[ Q = \int_0^{2\pi} \int_0^a w r \, dr \, d\theta = \frac{\pi Ga^4}{8\mu} \]

The mean flow velocity is
\[ w_{\text{av}} = \frac{\text{Flux}}{\text{Area of cross section}} = \frac{Ga^2}{8\mu} \]

**Problem 2 (Poiseuille Flow: Elliptic cross section).** Let us assume the axial velocity of the form
\[ w(x, y) = \alpha \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \]

It satisfies the boundary condition. Direct substitution into the governing equations for $w$ gives
\[ \alpha = -\frac{G}{2\mu} \frac{a^2}{a^2 + b^2} \]

Thus the velocity profile is given by
\[ w(x, y) = \frac{G}{2\mu} \frac{a^2}{a^2 + b^2} \left( 1 - \left[ \frac{x^2}{a^2} + \frac{y^2}{b^2} \right] \right) \]

**Problem 3 (Poiseuille Flow: Rectangular cross section).** Let the flow is confined within the channel given by $-a \leq x \leq a$ and $-b \leq y \leq b$. Thus the equation to be solved is
\[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{G}{\mu} \]

subject to
\[ w(\pm a) = w(\pm b) = 0 \]

We look for solution in the separable form i.e. $w(x, y) = X(x)Y(y)$ with $X(\pm a) = Y(\pm b) = 0$. Hence we choose $w$ as
\[ w(x, y) = \sum_{m=0}^\infty \sum_{n=0}^\infty A_{mn} \cos \left[ (2m + 1) \frac{\pi x}{2a} \right] \cos \left[ (2n + 1) \frac{\pi y}{2b} \right] \]

Where the constants $A_{mn}$ are to be found. These can be found by substituting into the governing equations for $w$. 
21.3 Flow down an inclined plane

Consider two dimensional flow of a fluid of height $h$ down an inclined plane. Fix the coordinate axes as shown in the figure. The boundary condition is that $p = p_0$ at $x = h$ and no-slip at $x = 0$.

We seek a steady solution of the $w = w(x)$ which satisfy the equation of continuity identically. The momentum equations are

$$0 = \frac{\partial p}{\partial x} - \rho g \sin \alpha$$
$$0 = -\frac{\partial p}{\partial z} + \rho g \cos \alpha + \mu \frac{d^2 w}{dx^2}$$

From the first equation we get

$$p = -\rho g \sin \alpha x + f(z)$$

Now using $p(x = h, z) = p_0$, we get $f(z) = p_a + \rho g \sin \alpha h$ and hence

$$p = p_0 + \rho g \sin \alpha (h - x)$$

Substituting into the equation for $w$ we get

$$\frac{d^2 w}{dx^2} = -\frac{\rho g \cos \alpha}{\mu}$$

Integrating twice we get

$$w = -\frac{\rho g \cos \alpha}{2\mu} x^2 + Ax + B$$

No-slip at $x = 0$ implies $B = 0$. If we neglect surface tension, then the stress vector at $x = h$ for the fluid and air will be same. Thus we have

$$\sigma_{ij}^{air} n_j = \sigma_{ij} n_j, \quad \mathbf{n} = (1, 0, 0), i = 1, 2, 3$$

For $i = 1$ we get

$$\sigma_{11}^{air} = \sigma_{11} \quad \text{or} \quad p_0 + 2\mu^{air} D_{11}^{air} = -p + 2\mu D_{11}$$
which is satisfied identically. For $i = 2$ we get
\[ \sigma_{21}^{air} = \sigma_{21} \quad \text{or} \quad 2 \mu^{air} D_{21}^{air} = 2 \mu D_{21} \]
This is also satisfied identically. For $i = 3$ we have
\[ \sigma_{31}^{air} = \sigma_{31} \quad \text{or} \quad 2 \mu^{air} D_{31}^{air} = 2 \mu D_{31} \]
Since $\mu^{air} \ll \mu$ we have at $x = h$
\[ D_{31} = 0 \quad \text{or} \quad \frac{dw}{dx} = 0 \quad \text{at} \quad x = h \]
This we used to get $A$ and the solution $w$ becomes
\[ w = \frac{pg \cos \alpha}{2\mu} x(2h - x) \]
The above solution can be written as
\[ w = \frac{pg \cos \alpha}{2\mu} (h^2 - (h - x)^2) \]
From this we derive
\[ w_{max} = \frac{pg \cos \alpha h^2}{2\mu} \quad \text{and} \quad Q = \frac{pg \cos \alpha h^3}{3\mu} \]

22 Exact Navier Stokes in cylindrical coordinate

We can make considerbale simplication by assuming that the flow is axisymmetric. This means that nothing changes with $\theta$ thus $\partial / \partial \theta = 0$. For an axisymmetric flow with no swirl we have
\[ v = u(r, z, t)e_r + w(r, z, t)e_z \]
and for an axisymmetric flow with swirl we have
\[ v = u(r, z, t)e_r + v(r, z, t)e_\theta + w(r, z, t)e_z \]

Problem (Steady axisymmetric swirl flow between coaxial cylinder). Fluid fills the annular gap between two infinitely long coaxial cylinder. The angular velocity of the inner/outer cylinders of radii $a$ and $b$ are $\Omega_1$ and $\Omega_2$ respectively. Since the swirl flow is steady and the cylinders are infiniely long we have
\[ v = v(r)e_\theta \]
The equation of continuity is automatically satisfied. The equations of motion are
\[ \frac{v^2}{r} = -\frac{1}{\rho} \frac{dp}{dr} \]
\[ 0 = \frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{v}{r^2} \]
Solving for $v$ gives

$$v = Ar + B/r$$

The boundary conditions are $v = a\Omega_1$ at $r = a$ and $v = b\Omega_2$ at $r = b$. Using these we get

$$A = \frac{b^2\Omega_2 - a^2\Omega_1}{b^2 - a^2} \quad b = a^2b^2\Omega_2 - \Omega_1$$

We also have for vorticity

$$\omega = \nabla \wedge v = 2\frac{b^2\Omega_2 - a^2\Omega_1}{b^2 - a^2} e_z$$

and is therefore constant. Now let us calculate the torque on the inner cylinder. To do this first we calculate the $D_{\theta r}$ which using the matrix of $\nabla v$ we get

$$D_{\theta r} = -\frac{B}{r^2}$$

Hence we get

$$\sigma_{\theta r} = -2\mu B/r^2$$

Now $\mathbf{n} = e_r$ and we have

$$\mathbf{t}_n = \sigma e_r = \sigma_{rr} e_r + \sigma_{\theta r} e_\theta + \sigma_{rz} e_z$$

We are interested horizontal torque per unit vertical length. This will be given by azimutnal component $\sigma_{\theta r}$. Now for the inner cylinder this torque is given by

$$\tau_a = \int_0^{2\pi} (a \sigma_{\theta r}|_{r=a}) a \, d\theta = -4\pi\mu B$$

Substituting the value of $a$ we get

$$\tau_a = -4\pi\mu a^2 b^2 \frac{\Omega_2 - \Omega_1}{b^2 - a^2}$$

Similar calculation for the outer cylinder gives

$$\tau_b = -4\pi\mu a^2 b^2 \frac{\Omega_2 - \Omega_1}{b^2 - a^2}$$

**Special cases**

Equal rotation rate: In this case the $\Omega_1 = \Omega_2$ and in this case we have

$$v = \Omega_1 r$$

Thus the fluid swirls as if it is a rotating rigid body and the couple on both the cylinder vanish.
No inner cylinder: In this case \( a = 0 \) and we must have \( B = 0 \) to remove the singularity. Again we have just a rigid body rotation

\[
v = r \Omega_2
\]

Stationary outer cylinder: We have \( \Omega_2 = 0 \) and we have

\[
\frac{v}{r} = \frac{a^2 \Omega_1}{b^2 - a^2 \left( \frac{b^2}{r^2} - 1 \right)}
\]

Letting \( b \to \infty \) we get

\[
v = \frac{a^2 \Omega_1}{r}
\]

## 23 Unsteady unidirectional flow

For unsteady unidirectional flow we have to solve

\[
v = we_3, \quad w = w(x,y,t), \quad \frac{\partial v}{\partial t} = -\frac{\nabla p}{\rho} + \nu \nabla^2 v
\]

As in the steady cases, we have

\[
\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \implies p = p(z,t)
\]

### 23.1 Impulsively started plate

Consider a semi-infinite fluid resting above a rigid plane. For \( t < 0 \) the fluid is at rest. At \( t = 0 \) the plane wall instanteneously acquires a speed \( U \) in the \( z \) direction. Considering the
figure, it is clear that the flow variables must be independent of \( x \) and hence \( w = w(y, t) \).

Now the momentum equation along \( z \) direction gives

\[
\rho \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial y^2}
\]

Now the above equation implying \( \partial p/\partial z \) must be independent of \( z \). Thus we have

\[
p(y, z, t) = p_0(y, t) + p_1(y, t)z
\]

Since pressure is boundend at \( z = \pm \infty \) we must have \( p_1(y, t) = 0 \) and hence \( \partial p/\partial z = 0 \)

Hence the equation reduces to

\[
\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial y^2}
\]

subject to the initial and boundary conditions

\[
w(0, t) = U \\
w(y, t) \to 0 \text{ as } y \to \infty \\
w(y, 0) = 0
\]

We get rid of the constant \( U \) by defining \( W = Uw \) thus reducing the problem to

\[
\frac{\partial W}{\partial t} = \nu \frac{\partial^2 W}{\partial y^2}
\]

subject to the initial and boundary conditions

\[
W(0, t) = 1 \\
W(y, t) \to 0 \text{ as } y \to \infty \\
W(y, 0) = 0
\]

Clearly we must have \( W = \phi(\nu, y, t) \).

Solution by Laplace transform: Taking Laplace transform on both sides we get

\[
s\tilde{W} = \nu \frac{d^2\tilde{W}}{\partial y^2}
\]

subject to

\[
\tilde{W}(0, s) = \frac{1}{s} \\
\tilde{W}'(y, s) \to 0 \text{ as } y \to \infty
\]

The solution is

\[
\tilde{W} = Ae^{y\sqrt{s/\nu}} + Be^{-y\sqrt{s/\nu}}
\]
Using the boundary conditions we get
\[ \tilde{W} = \frac{e^{-y\sqrt{s/\nu}}}{s} \]

The inverse transform gives
\[ W(y, t) = \text{erfc}(\eta), \quad \text{where} \quad \eta = \frac{y}{2\sqrt{\nu t}} \]

where \( \text{erfc}(\eta) \) is defined by
\[ \text{erfc}(\eta) = 1 - 2\sqrt{\pi} \int_{0}^{\eta} e^{-t^2} dt = 1 - \text{erf}(\eta) \]

Thus the solution is given by
\[ u = U \text{erfc}(\eta), \quad \eta = \frac{y}{2\sqrt{\nu t}} \]

Solution by Similarity: Since \( W \) is nondimensional and \( W = \phi(y, \nu, t) \), the function \( \phi \) must also be nondimensional. Since \( y, \nu, t \) involve two independent dimension \([L]\) and \([T]\) only one independent nondimensional variable is possible. Let us take the nondimensional variable as
\[ \eta = \frac{y}{2\sqrt{\nu t}} \]

Thus let us choose
\[ W = f(\eta) \]

Putting into the governing equations we have
\[ f'' = -2\eta f' \]

The boundary conditions are
\[ f(\infty) = 0 \quad f(0) = 1 \]

We solve the above as
\[ f' = Ae^{-\eta^2} \]

Integrating once again we get
\[ f = B + A \int_{0}^{\eta} e^{-\eta^2} d\eta \]

Putting into boundary conditions we get \( B = 1 \) and \( A = -2/\sqrt{\pi} \). Thus we have the same solution.
If we plot the \( u/U \) versus \( \eta \) then we observe that \( u/U = 0.05 \) at \( \eta \approx 1.4 \). Thus the width of the boundary layer \( \delta \approx 2.8\sqrt{\nu t} \). Thus the boundary layer (see later in the course) increases as \( \sqrt{t} \). The vorticity is given by

\[
\omega = \nabla \wedge \mathbf{v} = -\frac{U}{\pi \sqrt{\nu t}} e^{-\frac{y^2}{4\nu t}} e_1
\]

Using Gamma function we can show that

\[
\int_0^\infty \omega \, dy
\]

is a constant vector for \( t > 0 \). Thus there is no new vorticity generated in the flow. The vorticity at \( t = 0 \) due to impulsive start is simply diffuse away with time. And the vorticity at time \( t \) is confined within the layer \([0, 2\sqrt{\nu t}]\).

### 23.2 Oscillating flows

Consider the flow similar to the above but the plate is now oscillating with frequency \( \omega \). The governing equations are

\[
\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial y^2}
\]

subject to the boundary conditions

\[
\begin{align*}
w(0, t) &= U \cos(\omega t) \\
w(y, t) &\to 0 \text{ as } y \to \infty
\end{align*}
\]

We can remove the dependence on \( U \) by defining \( w = U W \). Thus reduces the problem to

\[
\frac{\partial W}{\partial t} = \nu \frac{\partial^2 W}{\partial y^2}
\]

subject to the boundary conditions

\[
\begin{align*}
W(0, t) &= \cos(\omega t) \\
W(y, t) &\to 0 \text{ as } y \to \infty
\end{align*}
\]

Note that here \( W = \phi(\nu, y, t, \omega) \) and \( \phi \) must be function of two independent nondimensional variable. Thus similarity solution is not possible. Since the frequency of the wall is \( \omega \) and the problem is linear, we assume that the frequency of the fluid flow \( w \) is also \( \omega \). To solve let us assume that

\[
W = \text{Re}(f(y) e^{i\omega t})
\]

Substituting into the governing equations we get

\[
f'' = \frac{i\omega}{\nu} f
\]
subject to the boundary conditions
\[ f(0) = 1 \quad \text{and} \quad f(y) \to 0 \quad \text{as} \quad y \to \infty \]
Thus we have
\[ f = Ae^{\lambda y} + Be^{-\lambda y}, \quad \lambda = \sqrt{\frac{i\omega}{\nu}} \]
Now \( \sqrt{i} = (1 + i)/\sqrt{2} \) we must have \( A = 0 \). Using the other conditions gives \( B = 1 \).
Thus we have
\[ f = e^{-\sqrt{\omega/2\nu}(1+i)y} \]
Hence
\[ w = U e^{-\sqrt{\omega/2\nu}y} \cos \left( \omega t - \sqrt{\omega/2\nu}y \right) \]
The vorticity is given by
\[ \omega = \nabla \times v = \frac{\partial w}{\partial y} e_1 = -U e^{-\sqrt{\omega/2\nu}y} \cos \left( \omega t - \sqrt{\omega/2\nu}y + \frac{\pi}{4} \right) e_1 \]
Thus most of the vorticity is confined within the region of thickness \( \sqrt{\nu/\omega} \). The higher the frequency the thinner the layer. The wall shear stress is proportional to
\[ \left. \frac{\partial w}{\partial y} \right|_{y=0} \propto \cos(\omega t) - \sin(\omega t) = \sqrt{2} \cos(\omega t + \pi/4) \]
Thus there is a phase difference of \( \pi/4 \) between the wall velocity and wall shear.

24 Stagnation point flow

Consider the flow of a viscous fluid towards the wall as shown in the figure. Introducing stream function we can write
\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]
If we considered the fluid to be inviscid and irrotational, then the governing equations for \( \psi \) is given by

\[
\nabla^2 \psi = 0
\]

subject to zero normal velocity at \( y = 0 \) i.e. \( v = 0 \) at \( y = 0 \) or equivalently \( \psi = 0 \) at \( y = 0 \).

The solution is

\[
\psi = kxy
\]

where \( k \) is a constant. Hence

\[
u = kx, \quad v = -ky
\]

Clearly this equation do not satisfy the noslip condition at \( y = 0 \). Hence let us assume solution of the inviscid form as

\[
v = -f(y), \quad u = xf'(y)
\]

This implies that \( f' \to k \) as \( y \to \infty \). With this choice the mass conservation equation is satisfied automatically. Now consider the \( y \)-momentum equation first. Putting the values of \( u \) and \( v \) we get

\[
p_y = -\rho(f f' + f'') = Q(y)
\]

Hence

\[
p = \bar{p}(x) + R(y)
\]

From the \( x \)-momentum equation we get

\[
p_x = \rho x(\nu f''' + ff'' - f'^2)
\]

Thus

\[
p = \frac{\rho x^2}{2} (\nu f''' + ff'' - f'^2) + S(y)
\]

The two form for \( p \) implies that

\[
\nu f''' + ff'' - f'^2 = \beta
\]

where \( \beta \) is a constant. Taking \( y \to \infty \) we get \( \beta = -k^2 \). Also the boundary conditions at \( y = 0 \) becomes \( f(0) = f'(0) = 0 \). Thus the complete problem is given by

\[
u f''' + ff'' + k^2 - f'^2 = 0
\]

subject to

\[
f(0) = f'(0) = 0 \quad \text{at} \quad y = 0, \quad f' \to k \quad \text{as} \quad y \to \infty
\]

The solution for the flow must be found numerically.