Stochastic User Equilibrium

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Abstract

User Equilibrium Models are used to analyse a network with given O-D matrix to find out the flows in each link that is a part of the network. This problem has been studied at different levels of comprehension ranging from deterministic, probabilistic, as well as dynamic models. This report gives a brief on the methods that are part of a group represented by *Stochastic User Equilibrium* (commonly referred to as *SUE*). In addition, a detailed description of DIAL's probabilistic model is provided that has provided the foundation for other SUE models.

1 Introduction

The traffic assignment problem has intrigued many researchers to develop models that can appropriately reflect the thought process that a driver faces while choosing way towards his/her destination. Most basic models among these are *Deterministic User Equilibrium* models. These models are based on Wardrop's First Principle which states that "The journey times in all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route" [1]). Each user non-cooperatively seeks to minimize his costs of transportation. The traffic flows which follows this principle will lead to a *User Equilibrium* point (commonly referred to as UE). This is called an equilibrium because no user can improve upon his costs from this point by unilaterally shifting from one route to another. The simplest route choice and assignment method is *All-or-Nothing* which assumes that there are no congestion effects and all drivers perceive the alternatives in the same way. An improvement over this method is an *Incremental Assignment Technique* in which the network is loaded incrementally using All-or-Nothing and the results of this assignment are used to update the travel time on the network as an input to another All-or-Nothing assignment. This is done iteratively to distribute all the flows in the network. These methods are deterministic in nature and assume that the drivers are perfectly rational and identical and that they have complete knowledge of the network and flows. This is usually not the case in reality and this is the reason why these models fall short in giving a proper estimate for any arbitrary network.

Even though the methods mentioned in UE are not complete in determining the behaviourial dynamics of the drivers in route choice, they give a good macroscopic nature of transportation sytem design. They help in framing models that can more correctly approximate the decision making process of the driver making a route choice. One class of methods that tries to do multipath assignment is *Stochastic User Equilibrium.* These models aim to model the variations in driver perceptions and are flexible enough to allow drivers to choose routes based on their different perceptions. Unlike deterministic UE, these models do not assume the drivers to have complete knowledge of the network conditions. Such models assign trips to alternative paths by assigning probabilities. These probabilities represent the likelihood of moving from one node to another using a particular link. What is good in these models is that multiple paths are considered at one trip assignment and trips are assigned on the basis of their goodness. This report focuses on one such method proposed by [2].

2 Probabilistic Assignment Model

This method considers multiple paths while assigning trips from one node to another. These multiple paths are categorized into two classes: *reasonable paths* and *unreasonable paths*. A reasonable path (also called as efficient path) is one that does not backtrack. In other words when one moves on that path he should always feel that he is going farther from the origin and closer to the destination. This leads to a more explicit definition of a reasonable path as one in which each link has its initial node closer to the origin than its final node and the final node closer to the destination than its initial node. The trips that this model is going to distribute will always follow efficient paths. It may happen that once in a while some trips follow inefficient paths but the share of such flows (as ratio of total flow) is very less and hence does not affect the trip distribution much. Following are the requirements (or assumptions) that this model follows:

1. For a given origin-destination, all reasonable paths should be given a nonzero probability of use whereas all unreasonable paths should be given zero probability.

2. All reasonable paths of equal length should have the same probability of use.

3. When there are two reasonable paths of unequal length, a higher probability of use should be assigned to the path with lower length.

4. User should have some control on path diversion probabilities.

5. The assignment algorithm should not explicitly enumerate paths.

The requirement (1) is analogous to the one in deterministic UE where only one path is chosen on the basis of it being the shortest as compared to all other paths. Here multiple paths (including the shortest path) are chosen and assigned probabilities of use according to their length (postulated in requirement (2) and (3)). The good thing about assigning probability of use rather than the flows itself is that we allow the degree of freedom in the algorithm to account for random decisions taken by drivers (due to their incomplete knowledge or non quantifiable preferences). The algorithm takes a user defined parameter θ, a positive number which can raise or lower diversion probabilities (requirement (4)). Say from a given origin-destination, the shortest path has length t^* and a reasonable path is chosen which has length t, then the probability that this path is chosen over the shortest path is $e^{\theta(t^*-t)}$. The probability of choosing a particular path that is δt length longer than the shortest path is given by $e^{-\theta\delta t}$. As θ increases the probability that the shortest path is chosen increases. When θ is zero, either path is good and this leads to all efficient paths to be considered equally in traffic assignment. Whereas when θ is very large, the pick would get biased towards the shortest path. Such an assignment will have equivalence with the All-or-Nothing assignment. Therefore this user defined parameter gives the flexibility to user to alter the choice made through probabilities of use. The model does not enumerate the paths it load, all efficient paths between an origin and destination are loaded simultaneously.

The algorithm for the probability assignment model is elaborated as below:
Algorithm 1: multipath probabilistic assignment

1. To assign y trips from origin o to destination d, we need p(i), the shortest distance path from o to i; q(i), the shortest distance path from i to d; I_i, the set of all links whose initial node is i; F_i, the set of all links whose final node is i. Assuming that link e = (i,j) has length t(i,j), the likelihood of choosing this link will be:

$$a(e) = \begin{cases} e^{\theta[p(j)-p(i)-t(i,j)]} & \text{if} p(i) < p(j), q(j) < q(i) \\ 0 & \text{otherwise} \end{cases}$$

   The algorithm will concern itself only with non zero values of a(e). It is divided into two steps: *Forward Pass* and *Backward Pass.*

2. (Forward Pass). By examining all nodes i in *ascending* sequence with respect to p(i), for each link e in I_i it's link weight is calculated as:

$$w(e) = \begin{cases} a(e) & \text{if } i = 0 \text{ (the origin)} \\ a(e)\sum_{e' in F_i} w(e') & \text{otherwise} \end{cases}$$

   When the node d is reached (which is the destination), *backward pass* is implemented.

3. (Backward Pass). Starting with the destination node d, all nodes j are examined in *descending* sequence with respect to p(j) (distance from origin of j^{th} node). Assign the trip volume x(e) to each link e in F_j as follows:

$$x(e) = \begin{cases} \frac{y.w(e)}{\sum_{e' in F_j} w(e')} & \text{if } j = d \text{ (the destination node)} \\ \frac{w(e)\sum_{e' in I_j} x(e')}{\sum_{e' in F_j} w(e')} & \text{otherwise} \end{cases}$$

   when the origin o is reached, stop. The assignment is complete.

In the first step of the algorithm, the likelihood of every link $e = (i, j)$ is defined using a(e). The argument of the exponential function in a(e) is proportional to $p(j)-(p(i)-t(i,j))$ which is (non-positive) difference between the shortest distance to node j and the shortest distance to node j which uses link (i,j). The probability of using any path P can be represented as product of likelihoods of links that are part of the path P. In the forward pass the link weights are calculated for all the links in the network that exists on paths that are efficient for a particular origin-destination. Then these link weights are used to determine the flows in the backward pass. In the backward pass, starting from the destination node d, y trips are distributed proportionately to the ratio of link weights that terminate to the node d. And for any node other than d, the total number of trips that needs to be assigned is first calculated by finding the total number of trips in all links that start at node j, and then distributing these trips to all links that terminate at node j proportionately to their link weights. It is worth mentioning that the calculation of functions x and w for all the links is done recursively.

3 Parallel Multipath Probabilistic Assignment

The algorithm 1 (probabilistic assignment model) requires one execution of its assignment algorithm for each distinct origin-destination pairs to determine the total flows in the links of the network. For a big network consisting of high number of nodes, the computational effort required by this algorithm would be high and requires more computationally effective model. An alternative procedure is proposed by [2] in which the definition for the efficient paths is altered where a path is efficient if and only if for all links that are part of this path has its initial node closer to the origin that its final node. This leads to increase in number of efficient paths (few of those paths which were not efficient due to their final node not being closer to the destination than its initial node). Thus trips will be spread over more links than before. Changing the definition of efficient paths leads to saving in computational effort. Even though the number of paths has increased, the assignment of trips is done faster as compared to algorithm 1. This procedure simultaneously assigns all trips starting from an origin node to all efficient paths originating from this node in one single execution whereas in algorithm 1, a single execution would give flows for a given origin-destination pair only. This difference in computational execution will become more clear as the algorithm for parallel multipath is elaborated as follows:
Algorithm 2: Parallel Multipath Probabilistic Assignment

1. To simultaneously assign all trips from origin o to all the destination nodes (represented by index i), four things are required: y(i), the number of trips from node o terminating at node i; p(i), p(i), shortest distance path from o to i; q(i), I_i, the set of all links whose initial node is i; F_i, the set of all links whose final node is i. Assuming that link e = (i,j) has length t(i,j), the likelihood of choosing this link will be:

$$a(e) = \begin{cases} e^{\theta[p(j)-p(i)-t(i,j)]} & \text{if } p(i) < p(j) \\ 0 & \text{otherwise} \end{cases}$$

   Notice that the constraint $q(j) < q(i)$ has been dropped since the definition of efficient path had been relaxed to include those paths which have $q(j) > q(i)$. Again the next two steps are divided into: *Forward Pass* and *Backward Pass.*

2. (Forward Pass). By examining all nodes i in *ascending* sequence with respect to p(i), for each link e in I_i it's link weight is calculated as:

$$w(e) = \begin{cases} a(e) & \text{if } i = 0 \text{ (the origin)} \\ a(e) \sum_{e' inF_i} w(e') & \text{otherwise} \end{cases}$$

   When the node d is reached (which is the destination), *backward pass* is implemented.

3. (Backward Pass). Starting with the destination node d, all nodes j are examined in *descending* sequence with respect to p(j) (distance from origin of j^{th} node). Assign the trip volume x(e) to each link e in F_j as follows:

   - Assign a trip volume x(e) to each link e using:

$$x(e) = \frac{y(j).w(e)}{\sum_{e' inF_j} w(e')}$$

- Increase the node volume at e's initial node i by e's link volume

$$y(i) = y(i) + x(e)$$

When the origin node o is reached, stop. The assignment is complete. All trips originating at node o has been assigned.

4 Proof of Concept: Algorithm1

The probability of using any path P is directly proportional to the product of likelihood of the links in the path, i.e.,

$$prob(P) = k \prod_{\text{e in P}} a(e)$$

This probability will be non-zero if and only if the path P is efficient. (if the path is inefficient, then one of the link will have zero likelihood and hence gives zero probability). This satisfies the *requirement 1*. Also we know the expression of a(e) (coming from the logit model). This expression when substituted in the prob(P) gives:

$$\begin{aligned} prob(P) &= k \prod_{\text{(i,j) in P}} e^{\theta[p(j)-p(i)-t(i,j)]} \\ &= ke^{\theta} \sum_{\text{(i,j) in P}} [p(j) - p(i) - t(i,j)] \\ &= ke^{\theta}[p(d) - \sum_{\text{(i,j) in P}} t(i,j)] \end{aligned}$$

The terms cancel in the expression to give p(d) as the shortest distance from origin o to destination d, and $\sum t(i,j)$ as the length of the path P. Since P may not be the shortest path connecting o-d, therefore the argument of the exponent will be negative and will decrease as the length of the path deviates more from the shortest distance. This means that two paths equally bad with respect to the shortest distance path are treated equally and for two paths with unequal lengths, the one with shorter path length will have more probability of getting picked over the another. This confirms the satisfaction of *requirement 2*.

5 Conclusions

The report tries to distinguish the characteristics of deterministic UE and SUE by highlighting the principles and assumptions these models follow. The prospects and consequences of either approaches are studied and the importance of SUE in modelling the route choice in a more effective way is presented. Dial's methods of probabilistic assignment is studied in detail which assigns weights to links in the forward pass and assigns traffic in backward pass. Proof of concept is provided in the end that justifies all the assumptions that are made in formulation of Algorithm1 satisfying the requirements of the model.

References

[1] J. Wardrop, (1952).Some theoretical aspects of road traffic research. *Proceedings of the institute of Civil Engineers*, 2(1): 325-378

[2] R. B. Dial, (1971). A Probabilistic Multipath Traffic Assignment Model Which Obviates Path Enumeration. *Transportation Research, 27B*, 61-75.