Analysis of Variance

LECTURE - 4

FACTORIAL EXPERIMENTS

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2^3 Factorial experiment

Suppose that in a complete factorial experiment, there are three factors - *A*, *B* and C, each at two levels, viz., $a_0, a_1; b_0, b_1$ and c_0, c_1 respectively. There are total eight number of combinations: $N = 2^3 = 8$

$$a_0b_0c_0,\ a_0b_0c_1,\ a_0b_1c_0,\ a_0b_1c_1,\ a_1b_0c_0,\ a_1b_0c_1,\ a_1b_1c_0,\ a_1b_1c_1.$$

Each treatment combination has r replicates, so the total number of observations are $N = 2^3 r = 8r$ that are to be analyzed for their influence on the response.

Assume the total response values are

$$Y_* = \left[(1),\ \ a,\ \ b,\ \ ab,\ \ c,\ \ ac,\ \ bc,\ \ abc\right]'.$$

The response values can be arranged in a three-dimensional contingency table. The effects are determined by the linear contrasts

$$\ell'_{effect} Y_* = \ell'_{effect}\left((1),\ \ a,\ \ b,\ \ ab,\ \ c,\ \ ac,\ \ bc,\ \ abc\right)$$

using the following table:

Factorial effect	Treatment combinations							
	(1)	*a*	*b*	*ab*	*c*	*ac*	*bc*	*abc*
I	+	+	+	+	+	+	+	+
A	-	+	-	+	-	+	-	+
B	-	-	+	+	-	-	+	+
AB	+	-	-	+	+	-	-	+
C	-	-	-	-	+	+	+	+
AC	+	-	+	-	-	+	-	+
BC	+	+	-	-	-	-	+	+
ABC	-	+	+	-	+	-	-	+

Note that once few rows have been determined in this table, rest can be obtained by simple multiplication of the symbols.

For example, consider the column corresponding to a, we note that

A has + sign, B has – sign ,

so AB has – sign (=sign of A x sign of B).

Once AB has - sign, C has – sign then ABC has (sign of AB x sign of C) which is + sign and so on.

The first row is a basic element. With this $a = 1'Y_*$ can be computed where 1 is a column vector of all elements unity.

If other rows are multiplied with the first row, they stay unchanged (therefore we call it as identity and denoted as I).

Every other row has the same number of + and – signs.

If + is replaced by 1 and – is replaced by -1, we obtain the vectors of orthogonal contrasts with the norm $8(= 2^3)$.

If each row is multiplied by itself, we obtain I (first row). The product of any two rows leads to a different row in the table. For example

$$A.B = AB$$

$$AB.B = AB^2 = A$$

$$AC.BC = A.C^2 BB = AB.$$

The structure in the table helps in estimating the average effect.

For example, the average effect of A is

$$A = \frac{1}{4r}\left[(a)-(1)+(ab)-(b)+(ac)-(c)+(abc)-(bc)\right]$$

which has following explanation.

(i) Average effect of A at low level of B and C

$$\equiv (a_1 b_0 c_0) - (a_0 b_0 c_0)$$

$$\equiv \frac{\left[(a)-(1)\right]}{r}$$

(ii) Average effect of A at low level of B and low level of C

$$\equiv (a_1 b_1 c_0) - (a_0 b_1 c_0)$$

$$\equiv \frac{\left[(ab)-(b)\right]}{r}$$

(iii) Average effect of A at low level of B and high level of C

$$\equiv (a_1 b_0 c_1) - (a_0 b_0 c_1)$$

$$\equiv \frac{[(ac) - (c)]}{r}$$

(iv) Average effect of A at low level of B and C

$$\equiv (a_1 b_1 c_1) - (a_0 b_1 c_1)$$

$$\equiv \frac{[(abc) - (bc)]}{r}.$$

Hence for all combinations of B and C, the average effect of A is the average of all the average effects in (i)-(iv).

Similarly, other main and interaction effects are as follows:

$$B = \frac{1}{4r}\left[(b)+(ab)+(bc)+(abc)-(1)-(a)-(c)-(ac)\right] = \frac{(a+1)(b-1)(c+1)}{4r}$$

$$C = \frac{1}{4r}\left[c+(ac)+(bc)+(abc)-(1)-(a)-(b)-(ab)\right] = \frac{(a+1)(b+1)(c-1)}{4r}$$

$$AB = \frac{1}{4r}\left[(1)+(ab)+(c)+(abc)-(a)-(b)-(ac)-(bc)\right] = \frac{(a-1)(b-1)(c+1)}{4r}$$

$$AC = \frac{1}{4r}\left[(1)+(b)+(ac)+(abc)-(a)-(ab)-(c)-(bc)\right] = \frac{(a-1)(b+1)(c-1)}{4r}$$

$$BC = \frac{1}{4r}\left[(1)+(a)+(bc)+(abc)-(b)-(ab)-(c)-(ac)\right] = \frac{(a+1)(b-1)(c-1)}{4r}$$

$$ABC = \frac{1}{4r}\left[(abc)+(ab)+(b)+(c)-(ab)-(ac)-(bc)-(1)\right] = \frac{(a-1)(b-1)(c-1)}{4r}.$$

Various sum of squares in the 2^3 factorial experiment are obtained as

$$SS(Effect) = \frac{(\text{linear contrast})^2}{8r} = \frac{\left(\ell'_{effect} Y_*\right)^2}{r\ell'_{effect}\ell_{effect}}$$

which follow a Chi-square distribution with one degree of freedom under normality of Y_*. The corresponding mean squares are obtained as

$$MS(Effect) = \frac{SS(Effect)}{\text{Degrees of freedom}}.$$

The corresponding *F*-statistics are obtained by

$$F_{effect} = \frac{MS(Effect)}{MS(Error)}$$

which follows an *F*-distribution with degrees of freedoms 1 and error degrees of freedom under respective null hypothesis.

The decision rule is to reject the corresponding null hypothesis at α level of significance whenever

$$F_{effect} > F_{1-\alpha}(1, df_{\text{error}}).$$

These outcomes are presented in the following ANOVA table.

Sources	Sum of squares	Degrees of freedom	Mean squares	F
A	SSA	1	MSA	F_A
B	SSB	1	MSB	F_B
AB	$SSAB$	1	$MSAB$	F_{AB}
C	SSC	1	MSC	F_C
AC	$SSAC$	1	$MSAC$	F_{AC}
BC	$SSBC$	1	$MSBC$	F_{BC}
ABC	$SSABC$	1	$MSABC$	F_{ABC}
$Error$	$SS(Error)$	$8(r-1)$	$MS(Error)$	
Total	TSS	$8r - 1$		