Analysis of Variance

LECTURE - 5

FACTORIAL EXPERIMENTS

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2^n Factorial experiment

Based on the theory developed for 2^2 and 2^3 factorial experiments, we now extend them for 2^n factorial experiment.

- Capital letters $A, B, C, \ldots$ denote the **factors**. They are the main effect contrast for the factors $A, B, C, \ldots$
- $AB, AC, BC, \ldots$ denote the first order or **2-factor interactions**
- $ABC, ABD, BCD, \ldots$ denote the **second order** or **3-factor interactions** and so on.

- Each of the main effect and interaction effect carries one degree of freedom.
- Total number of main effects = $\binom{n}{1} = n.$
- Total number of first order interactions = $\binom{n}{2}$
- Total number of second order interactions = $\binom{n}{3}$

and so on.

Standard order for treatment combinations

The list of treatments can be expressed in a standard order.

- For one factor A, the standard order is (1), a.

- For two factors A and B, the standard order is obtained

  by adding b and ab in the standard order of one factor A.

  This is derived by multiplying (1) and a by b , i.e.

$$b \times \{(1), a\} = (1),\ a,\ b,\ ab.$$

Standard order for treatment combinations

- For three factors, add c, ac, bc and abc which are derived by multiplying the standard order of A and B by c, i.e.

$$c \times \{(1, \quad a, \quad b, \quad ab\} = (1), \quad a, \quad b, \quad ab, \quad c, \quad ac, \quad bc, \quad abc.$$

Standard order for treatment combinations

Thus the standard order of any factor is obtained step by step by multiplying it with additional letter to preceding standard order.

For example, the standard order of A, B, C and D is 2^4 factorial experiment is

$$(1),\ a,\ b,\ ab,\ c,\ ac,\ bc,\ abc,\ d\times\{(1),\ a,\ b,\ ab,\ c,\ ac,\ bc,\ abc\}$$
$$=(1),\ a,\ b,\ ab,\ c,\ ac,\ bc,\ abc,\ d,\ ad,\ bd,\ abd,\ cd,\ acd,\ bcd,\ abcd.$$

How to find the contrasts for main effects and interaction effect

For example, in a 2^2 factorial experiment, we had expressed

$$A = \frac{1}{2}(a-1)(b+1) = \frac{1}{2}\left[(-1)+(a)-(b)+(ab)\right]$$

$$AB = \frac{1}{2}(a-1)(b-1) = \frac{1}{2}\left[(1)-(a)-(b)+(ab)\right].$$

Note that each effect has two component - divisor and contrast.

When the order of factorial increases, it is cumbersome to derive such expressions.

First we detail how to write divisor and then illustrate the methods for obtaining the contrasts.

How to write divisor

In a 2^n factorial experiment , the

- general mean effect has divisor 2^n and
- any effect (main or interaction) has divisor 2^{n-1}.

How to write divisor

For example, in a 2^6 factorial experiment,

the general mean effect has divisor 2^6 and

any main effect or interaction effect of any order has divisor

$2^{6-1} = 2^5$.

If *r* replicates of each effect are available , then

- general mean effect has divisor $r\,2^n$ and
- any main effect or interaction effect of any order has divisor $r\,2^{n-1}$.

How to write contrasts

Contrast belonging to the main effects and the interaction effects are written as follows:

$$A = (a-1)(b+1)(c+1)...(z+1)$$

$$B = (a+1)(b-1)(c+1)...(z+1)$$

$$C = (a+1)(b+1)(c-1)...(z+1)$$

$$\vdots \qquad \qquad \vdots$$

$$AB = (a-1)(b-1)(c+1)...(z+1)$$

$$BC = (a+1)(b-1)(c-1)...(z+1)$$

$$\vdots \qquad \qquad \vdots$$

$$ABC = (a-1)(b-1)(c-1)...(z+1)$$

$$\vdots \qquad \qquad \vdots$$

$$ABC...Z = (a-1)(b-1)(c-1)...(z-1).$$

How to write contrasts

For example, in a 2^3 factorial

$$A = \frac{1}{2^{3-1}}(a-1)(b+1)(c+1)$$

$$= \frac{1}{4}\left[-(1)+(a)-(b)+(ab)-(c)+(ac)-(bc)+(abc)\right]$$

$$M = \frac{1}{2^{3}}(a+1)(b+1)(c+1)$$

$$= \frac{1}{8}\left[(1)+(a)+(b)+(ab)+(c)+(ac)+(bc)+(abc)\right]$$

Sums of squares

Suppose 2^n factorial experiment is carried out in a randomized block design with *r* replicates.

Denote the total yield (output) from *r* plots (experimental units) receiving a particular treatment combination by the same symbol within a square bracket.

For example, $[ab]$ denotes the total yield from the plots receiving the treatment combination (ab).

Sums of squares

In a 2^2 factorial experiment, the factorial effect totals are

$$[A]=[ab]-[b]+[a]-[1]$$

- $[ab]$ = treatment total, i.e. sum of r observations in which both the factors A and B are at second level.
- $[a]$ = treatment total, i.e., sum of r observations in which factor A is at second level and factor B is at first level.
- $[b]$ = treatment total, i.e., sum of r observations in which factor A is at first level and factor B is at second level.
- $[1]$ = treatment total, i.e., sum of r observations in which both the factors A and B are at first level.

Sums of squares $[A]=[ab]-[b]+[a]-[1]$

Thus

$$[A]=\sum_{i=1}^{r}\left[y_{i(ab)}-y_{i(b)}+y_{i(a)}-y_{i(1)}\right]$$
$$=\ell'_A y_A \text{ (say).}$$

where ℓ_A is a vector of +1 and -1 and y_A is a vector denoting the responses from ab, b, a and 1.

Similarly, other effects can also be found.

The sum of squares due to a particular effect is obtained as

$$\frac{[\text{Total yield}]^2}{\text{Total number of observations}}.$$

In a 2^2 factorial experiment in an RBD, the sum of squares due to A is

$$SSA = \frac{(\ell'_A y_A)^2}{r2^2}.$$

In a 2^n factorial experiment in an RBD, the divisor will be $r.\ 2^n$.

For example:

In 2^2 factorial experiment in an RBD with *r* replications, the division of degrees of freedom and the treatment sum of squares are as follows:

Source	Degrees of freedom	Sum of squares
Replications	r - 1	
Treatments	4 – 1 = 3	
A	1	$[A]^2/4r$
B	1	$[B]^2/4r$
AB	1	$[AB]^2/4r$
Error	3(r - 1)	
Total	4r - 1	

The decision rule is to reject the concerned null hypothesis when the related F - statistic $F_{effect} > F_{1-\alpha}(1, 3(r-1)).$