Chapter 10

Partial Confounding

The objective of confounding is to mix the less important treatment combinations with the block effect differences so that higher accuracy can be provided to the other important treatment comparisons. When such mixing of treatment contrasts and block differences is done in all the replicates, then it is termed as **total confounding.** On the other hand, when the treatment contrast is not confounded in all the replicates but only in some of the replicates, then it is said to be partially confounded with the blocks. It is also possible that one treatment combination is confounded in some of the replicates and another treatment combination is confounded in other replicates which are different from the earlier replicates. So the treatment combinations confounded in some of the replicates and unconfounded in other replicates. So the treatment combinations are said to be partially confounded. The partially confounded contrasts are estimated only from those replicates where it is not confounded. Since the variance of the contrast estimator is inversely proportional to the number of replicates in which they are estimable, so some factors on which information is available from all the replicates are more accurately determined.

Balanced and unbalanced partially confounded design

If all the effects of a certain order are confounded with incomplete block differences in equal number of replicates in a design, then the design is said to be balanced partially confounded design. If all the effects of a certain order are confounded an unequal number of times in a design, then the design is said to be unbalanced partially confounded design.

We discuss only the analysis of variance in the case of balanced partially confounded design through examples on 2^2 and 2^3 factorial experiments.

Example 1:

Consider the case of 2^2 factorial as in following table in the set up of a randomized block design.

Factorial effects	Treatment combinations				Divisor
	(1)	(a)	(b)	(ab)	
M	+	+	+	+	4
A	-	+	-	+	2
B	-	-	+	+	2
AB	+	-	-	+	2

where $y_{*i} = ((1), a, b, ab)'$ denotes the vector of total responses in the i^{th} replication and each treatment is replicated r times, $i = 1.2,...,r.$ If no factor is confounded then the factorial effects are estimated using all the replicates as

$$A = \frac{1}{2r}\sum_{i=1}^{r}\ell'_A y_{*i},$$
$$B = \frac{1}{2r}\sum_{i=1}^{r}\ell'_B y_{*i},$$
$$AB = \frac{1}{2r}\sum_{i=1}^{r}\ell'_{AB} y_{*i},$$

where the vectors of contrasts $\ell_A, \ell_B, \ell_{AB}$ are given by

$$\ell_A = (-1 \quad 1 \ -1 \ 1)'$$
$$\ell_B = (-1 \ -1 \quad 1 \ 1)'$$
$$\ell_{AB} = (1 \ -1 \ -1 \ 1)'.$$

We have in this case

$$\ell'_A\ell_A = \ell'_B\ell_B = \ell'_{AB}\ell_{AB} = 4.$$

The sum of squares due to A, B and AB can be accordingly modified and expressed as

$$SS_A = \frac{(\sum_{i=1}^{r}\ell'_A y_{*i})^2}{r\ell'_A\ell_A} = \frac{(ab + a - b - (1))^2}{4r}$$

$$SS_B = \frac{(\sum_{i=1}^{r}\ell'_B y_{*i})^2}{r\ell'_B\ell_B} = \frac{(ab + b - a - (1))^2}{4r}$$

and

$$S_{AB} = \frac{(\sum_{i=1}^{r}\ell'_{AB} y_{*i})^2}{r\ell'_{AB}\ell_{AB}} = \frac{(ab + (1) - a - b)^2}{4r},$$

respectively.

Now consider a situation with 3 replicates with each consisting of 2 incomplete blocks as in the following figure:

Replicate 1 (A confounded)		Replicate 2 (B confounded)		Replicate 3 (AB confounded)	
Block 1	Block 2	Block 1	Block 2	Block 1	Block 2
ab (1)	a b	ab a	b (1)	ab b	a (1)

There are three factors A, B and AB. In case of total confounding, a factor is confounded in all the replicates. We consider here the situation of partial confounding in which a factor is not confounded in all the replicates.

Rather, the factor A is confounded in replicate 1, the factor B is confounded in replicate 2 and the interaction AB is confounded in replicate 3. Suppose each of the three replicate is repeated r times. So the observations are now available on r repetitions of each of the blocks in the three replicates. The partitions of replications, the blocks within replicates and plots within blocks being randomized. Now from the setup of figure,

- the factor A can be estimated from replicates 2 and 3 as it is confounded in replicate 1.
- the factor B can be estimated from replicates 1 and 3 as it is confounded in replicate 2 and
- the interaction AB can be estimated from replicates 1 and 2 as it is confounded in replicate 3.

When A is estimated from the replicate 2 only, then its estimate is given by

$$A_{rep2} = \frac{(\sum_{i=1}^{r} \ell'_{A2} y_{*i})_{rep2}}{2r}$$

and when A is estimated from the replicate 3 only, then its estimate is given by

$$A_{rep3} = \frac{(\sum_{i=1}^{r} \ell'_{A3} y_{*i})_{rep3}}{2r}$$

where ℓ'_{A2} and ℓ'_{A3} are the suitable vectors of +1 and -1 for being the linear function to be contrasts under replicates 2 and 3, respectively. Note that ℓ'_{A2} *and* ℓ'_{A3} each is a (4×1) vectorhaving 4 elements in it. Now since A is estimated from both the replicates 2 and 3, so to combine them and to obtain a single estimator of A, we consider the arithmetic mean of A_{rep2} and A_{rep3} as an estimator of A given by

$$A_{pc} = \frac{A_{rep2} + A_{rep3}}{2}$$
$$= \frac{\left(\sum_{i=1}^{r} \ell'_{A2} y_{*1}\right)_{rep2} + \left(\sum_{i=1}^{r} \ell'_{A3} y_{*1}\right)_{rep3}}{4r}$$
$$= \frac{\sum_{i=1}^{r} \ell'_{A} y_{*1}}{4r}$$

where (8×1) the vector

$$\ell^{*'}_{A} = (\ell_{A2},\ \ell_{A3})$$

has 8 elements in it and subscript pc in A_{pc} denotes the estimate of A under "partial confounding" (pc). The sum of squares under partial confounding in this case is obtained as

$$SS_{Apc} = \frac{(\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i})^2}{r\ell^{*'}_{A}\ell^{*}_{A}} = \frac{(\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i})^2}{8r}.$$

Assuming that y_{ij}'s are independent and $Var(y_{ij}) = \sigma^2$ for all i and j, the variance of A_{pc} is given by

$$Var(A_{pc}) = \left(\frac{1}{4r}\right)^2 Var\left(\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i}\right)$$
$$= \left(\frac{1}{4r}\right)^2 Var\left((\sum_{i=1}^{r} \ell'_{A2} y_{*i})_{rep2} + (\sum_{i=1}^{r} \ell'_{A3} y_{*i})_{rep3}\right)$$
$$= \left(\frac{1}{4r}\right)^2 \left(4r\sigma^2 + 4r\sigma^2\right)$$
$$= \frac{\sigma^2}{2r}.$$

Now suppose A is not confounded in any of the blocks in the three replicates in this example. Then A can be estimated from all the three replicates, each repeated r times. Under such a condition, an estimate of A can be obtained using the same approach as the arithmetic mean of the estimates obtained from each of the three replicates as

$$A^{*}_{pc} = \frac{A_{rep1} + A_{rep2} + A_{rep3}}{3}$$
$$= \frac{(\sum_{i=1}^{r} \ell'_{A1} y_{*i})_{rep1} + (\sum_{i=1}^{r} \ell'_{A2} y_{*i})_{rep2} + (\sum_{i=1}^{r} \ell'_{A3} y_{*i})_{rep3}}{6r}$$
$$= \frac{\sum_{i=1}^{r} \ell^{**'}_{A} y_{*i}}{6r}$$

where (12×1) the vector

$$\ell_A^{**'} = (\ell_{A1}, \ \ell_{A2}, \ \ell_{A3})$$

has 12 elements in it. The variance of A assuming that y_{ij}'s are independent and $Var(y_{ij}) = \sigma^{*2}$ for all i and j in this case is obtained as

$$\begin{aligned} Var(A_{pc}^*) &= \left(\frac{1}{6r}\right)^2 Var\left[\left(\sum_{i=1}^r \ell'_{A1} y_{*i}\right)_{rep1} + \left(\sum_{i=1}^r \ell'_{A2} y_{*i}\right)_{rep2} + \left(\sum_{i=1}^r \ell'_{A3} y_{*i}\right)_{rep3}\right] \\ &= \left(\frac{1}{6r}\right)^2 \left(4r\sigma^{*2} + 4r\sigma^{*2} + 4r\sigma^{*2}\right) \\ &= \frac{\sigma^{*2}}{3r}. \end{aligned}$$

If we compare this estimator with the earlier estimator for the situation where A is unconfounded in all the r replicates and was estimated by

$$A = \frac{\sum_{i=1}^r \ell'_A y_{*i}}{2r}$$

and in the present situation of partial confounding, the corresponding estimator of A is given by

$$A_{pc}^* = \frac{A_{rep1} + A_{rep2} + A_{rep3}}{3} = \frac{\sum_{i=1}^r \ell_A^{**'} y_{*i}}{6r}.$$

Both the estimators, viz., A and A_{pc}^* are same because A is based on r replications whereas A_{pc}^* is based on $3r$ replications. If we denote $r^* = 3r$ then A_{pc}^* becomes same as A. The expressions of variance of A and A_{pc}^* then also are same if we use $r^* = 3r$. Comparing them, we see that the information on A in the partially confounded scheme relative to that in unconfounded scheme is

$$\frac{2r/\sigma^2}{3r/\sigma^{*2}} = \frac{2}{3}\frac{\sigma^{*2}}{\sigma^2}.$$

If $\sigma^{*2} > \frac{3}{2}\sigma^2$, then the information in partially confounded design is more than the information in unconfounded design.

In total confounding case, the confounded effect is completely lost but in the case of partial confounding, some information about the confounded effect can be recovered. For example, two third of the total information can be recovered .his case for A.

Similarly, when B is estimated from the replicates 1 and 3 separately, then the individual estimates of B are given by

$$B_{rep1} = \frac{\left(\sum_{i=1}^{r} \ell'_{B1} y_{*i}\right)_{rep1}}{2r}$$

and

$$B_{rep3} = \frac{\left(\sum_{i=1}^{r} \ell'_{B3} y_{*i}\right)_{rep3}}{2r}.$$

Both the estimators are combined as arithmetic mean and the estimator of B based on partial confounding is

$$\begin{aligned} B_{pc} &= \frac{B_{rep1} + B_{rep3}}{2} \\ &= \frac{\left(\sum_{i=1}^{r} \ell'_{B1} y_{*i}\right)_{rep1} + \left(\sum_{i=1}^{r} \ell'_{B3} y_{*i}\right)_{rep3}}{4r} \\ &= \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{B} y_{*i}\right)}{4r} \end{aligned}$$

where the (8×1) vector

$$\ell^{*'}_{B} = \left(\ell_{B1}, \ \ell_{B3}\right)$$

has 8 elements. The sum of squares due to B_{pc} is obtained as

$$SS_{B_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{B} y_{*i}\right)^2}{r \ell^{*'}_{B} \ell^{*}_{B}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{B} y_{*i}\right)^2}{8r}.$$

Assuming that y_{ij}'s are independent and $Var(y_{ij}) = \sigma^2$, the variance of B_{pc} is

$$\begin{aligned} Var(B_{pc}) &= \left(\frac{1}{4r}\right)^2 Var\left(\sum_{i=1}^{r} \ell^{*'}_{B} y_{*i}\right) \\ &= \frac{\sigma^2}{2r}. \end{aligned}$$

When AB is estimated from the replicates 1 and 2 separately, then its estimators based on the observations available from replicates 1 and 2 are

$$AB_{rep1} = \frac{\left(\sum_{i=1}^{r} \ell'_{AB} y_{*i}\right)_{rep1}}{2r}$$

and

$$AB_{rep2} = \frac{\left(\sum_{i=1}^{r} \ell'_{AB2} y_{*1}\right)_{rep2}}{2r}$$

respectively. Both the estimators are combined as arithmetic mean and the estimator of AB is obtained as

$$\begin{aligned} AB_{pc} &= \frac{AB_{rep1} + AB_{rep2}}{2} \\ &= \frac{\left(\sum_{i=1}^{r} \ell'_{AB1} y_{*i}\right)_{rep1} + \left(\sum_{i=1}^{r} \ell'_{AB2} y_{*i}\right)_{rep2}}{4r} \\ &= \frac{\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}}{4r} \end{aligned}$$

Where (8×1) the vector

$$\ell^{*'}_{AB} = (\ell_{AB1}, \ \ell_{AB2})$$

consists of 8 elements. The sum of squares due to AB_{pc} is

$$\begin{aligned} SS_{AB_{pc}} &= \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}\right)^2}{r \ell^{*}_{AB} \ell^{*}_{AB}} \\ &= \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}\right)^2}{8r} \end{aligned}$$

and the variance of AB_{pc} under the assumption that y_{ij}'s are independent and $Var\left(y_{ij}\right) = \sigma^2$ is given by

$$\begin{aligned} Var(AB_{pc}) &= \left(\frac{1}{4r}\right)^2 Var\left(\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}\right) \\ &= \frac{\sigma^2}{2r}. \end{aligned}$$

Block sum of squares:

Note that is case of partial confounding, the block sum of squares will have two components – due to replicates and within replicates. So the usual sum of squares due to blocks need to be divided into two components based on these two variants.. Now we illustrate how the sum of squares due to blocks are adjusted under partial confounding. We consider the setup as in the earlier example. There are 6 blocks (2 blocks under each replicate 1, 2 and 3), each repeated r times. So there are total $(6r-1)$ degrees of freedom associated with the sum of squares due to blocks. The sum of squares due to blocks is divided into two parts-

- the sum of squares due to replicates with $(3r-1)$ degrees of freedom and
- the sum of squares due to within replicates with $3r$ degrees of freedom.

Now, denoting

- B_i to be the total of i^{th} block and
- R_i to be the total due to i^{th} replicate,

the sum of squares due to blocks is

$$
\begin{aligned}
SS_{Block(pc)} &= \frac{1}{\text{Total number of treatment}} \sum_{\text{i=1}}^{\text{Total number of blocks}} B_i^2 - \frac{G^2}{N} \\
&= \frac{1}{2^2}\sum_{i=1}^{3r} B_i^2 - \frac{G^2}{12r}; \ (N=12r) \\
&= \frac{1}{2^2}\sum_{i=1}^{3r} (B_i^2 - R_i^2 + R_i^2) - \frac{G^2}{12r} \\
&= \frac{1}{2^2}\sum_{i=1}^{3r} \left(B_i^2 - R_i^2\right) + \left(\frac{1}{2^2}\sum_{i=1}^{3r} R_i^2 - \frac{G^2}{12r}\right) \\
&= \frac{1}{2^2}\sum_{i=1}^{3r} \left(\frac{B_{1i}^2 + B_{2i}^2}{2} - R_i^2\right) + \left(\frac{1}{2^2}\sum_{i=1}^{3r} R_i^2 - \frac{G^2}{12r}\right)
\end{aligned}
$$

where B_{ji} denotes the total of j^{th} block in i^{th} replicate $(j=1,2)$. The sum of squares due to blocks within replications (*wr)* is

$$SS_{\text{Block}(wr)} = \frac{1}{2^2}\sum_{i=1}^{3r}\left(\frac{B_{1i}^2 + B_{2i}^2}{2} - R_i^2\right).$$

The sum of squares due to replications is

$$SS_{\text{Block}(r)} = \frac{1}{2^2}\sum_{i=1}^{3r} R_i^2 - \frac{G^2}{12r}.$$

So we have in case of partial confounding

$$SS_{\text{Block}} = SS_{\text{Block}(wr)} + SS_{\text{Block}(r)}.$$

The total sum of squares remains same as usual and is given by

$$SS_{\text{Total}(pc)} = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{G^2}{N}; \quad (N = 12r).$$

The analysis of variance table in this case of partial confounding is given in the following table. The test of hypothesis can be carried out in a usual way as in the case of factorial experiments.

Source	Sum of squares	Degrees of freedom	Mean squares
Replicates	$SS_{\text{Block(r)}}$	$3r(=r^*)$	$MS_{\text{Block}}(r)$
Blocks within replicates	$SS_{\text{Block}}(wr)$	$3r-1(=r^*-1)$	$MS_{\text{Block(wr)}}$
Factor A	$SS_{A_{pc}}$	1	$MS_{A(pc)}$
Factor B	$SS_{B_{pc}}$	1	$MS_{B(pc)}$
AB	$SS_{AB_{pc}}$	1	$MS_{AB(pc)}$
Error	By subtraction	$6r-3(=2r^*-3)$	$MS_{E(pc)}$
Total	$SS_{\text{Total}(pc)}$	$12r-1(=4r^*-1)$	

Example 2:

Consider the setup of 2^3 factorial experiment. The block size is 2^2 and 4 replications are made as in the following figure.

Replicate 1 (AB confounded)		Replicate 2 (AC confounded)		Replicate 3 (BC confounded)		Replicate 4 (ABC confounded)	
Block 1	Block 2	Block 1	Block 2	Block 1	Block 2	Block 1	Block 2
(1)	a	(1)	a	(1)	b	(1)	a
ab	b	b	ab	a	c	ab	b
c	ac	ac	c	bc	ab	ac	c
abc	bc	abc	bc	abc	ac	bc	abc

The arrangement of the treatments in different blocks in various replicates is based on the fact that different interaction effects are confounded in the different replicates. The interaction effect AB is confounded in replicate 1, AC is confounded in replicate 2, BC is confounded in replicate 3 and ABC is confounded in replicate 4. Then the r replications of each block are obtained. There are total eight factors involved in this case including (1). Out of them, three factors, viz., A, B and C are unconfounded whereas AB, BC, AC and ABC are partially confounded. Our objective is to estimate all these factors. The unconfounded factors can be estimated from all the four replicates whereas partially confounded factors can be estimated from the following replicates:

- AB from the replicates 2, 3 and 4,
- AC from the replicates 1, 3 and 4,
- BC from the replicates 1, 2 and 4 and
- ABC from the replicates 1, 2 and 3.

We first consider the estimation of unconfounded factors A, B and C which are estimated from all the four replicates.

The estimation of factor A from ℓ^{th} replicate $(\ell = 1,2,3,4)$ is as follows:

$$A_{rep\, j} = \frac{\sum_{i=1}^{r} \ell'_{Aj} y_{*i}}{4r}$$

$$A = \frac{\sum_{j=1}^{4} \ell'_{Aj} A_{rep\, j}}{4} = \frac{\sum_{j=1}^{4}\sum_{i=1}^{r} \ell'_{Aj} y_{*i}}{16r}$$

$$= \frac{\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i}}{16r}$$

where (32×1) the vector

$$\ell^{*'}_{A} = \left(\ell_{A1},\ \ell_{A2},\ \ell_{A3},\ \ell_{A4}\right)$$

has 32 elements and each $\ell_{Aj}\ (j = 1,2,3,4)$ is having 8 elements in it. The sum of square due to A is now based on $32r$ elements as

$$SS_A = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i}\right)^2}{r\ell^{*'}_{A}\ell^{*}_{A}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i}\right)^2}{32r}.$$

Assuming that y_{ij}'s are independent and $Var(y_{ij}) = \sigma^2$ for all i and j, the variance of A is obtained as

$$\begin{aligned}Var(A) &= \left(\frac{1}{16r}\right)^2 Var\left(\sum_{i=1}^{r} \ell_A^{*'} y_{*i}\right)\\ &= \left(\frac{1}{16r}\right)^2 \times 32r\sigma^2\\ &= \frac{\sigma^2}{8r},\end{aligned}$$

Similarly, the factor B is estimated as an arithmetic mean of the estimates of B from each replicate as

$$B = \frac{\sum_{i=1}^{r} \ell_B^{*'} y_{*i}}{16r}$$

where (32×1) the vector

$$\ell_B^{*'} = (\ell_{B1},\ \ell_{B2},\ \ell_{B3},\ \ell_{B4})$$

consists of 32 elements.

The sum of squares due to B is obtained on the similar line as in case of A as

$$SS_B = \frac{\left(\sum_{i=1}^{r} \ell_B^{*'} y_{*i}\right)^2}{32r}.$$

The variance of B is obtained on the similar lines as in the case of A as

$$Var(B) = \frac{\sigma^2}{8r}.$$

The unconfounded factor C is also estimated as the average of estimates of C from all the replicates as

$$C = \frac{\sum_{i=1}^{r} \ell_C^{*'} y_{*i}}{16r}$$

where (32×1) the vector

$$\ell_C^{*'} = (\ell_{C1},\ \ell_{C2},\ \ell_{C3},\ \ell_{C4})$$

consists of 32 elements.

The sum of square due to C in this case is obtained as

$$SS_C = \frac{\left(\sum_{i=1}^{r} \ell_C^{*'} y_{*i}\right)^2}{32r}.$$

The variance of C is obtained as

$$Var(C) = \frac{\sigma^2}{8r}.$$

Next we consider the estimation of the confounded factor AB. This factor AB can be estimated from each of the replicates 2, 3 and 4 and the final estimate of AB can be obtained as the arithmetic mean of those three estimates as

$$\begin{aligned} AB_{pc} &= \frac{AB_{rep2} + AB_{rep3} + AB_{rep4}}{3} \\ &= \frac{1}{12r}\left(\left(\sum_{i=1}^{r} \ell'_{AB2} y_{*i})_{rep2}\right) + \left(\sum_{i=1}^{r} \ell'_{AB3} y_{*i})_{rep3}\right) + \left(\sum_{i=1}^{r} \ell'_{AB3} y_{*i})_{rep4}\right)\right) \\ &= \frac{\sum_{i=1}^{r} \ell'_{AB} y_{*i}}{12r} \end{aligned}$$

where (24×1) the vector

$$\ell^{*'}_{AB} = (\ell_{AB2},\ \ell_{AB3},\ \ell_{AB4})$$

consists of 24 elements and each of the (8×1) vectors ℓ_{AB2}, ℓ_{AB3} and ℓ_{AB4} is having 8 elements in it. The sum of squares due to AB_{pc} is then based on $24r$ elements given as

$$SS_{AB_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell^{*}_{AB'} y_{*i}\right)^2}{r \ell^{*}_{AB'} \ell^{*}_{AB}} = \frac{\left(\sum_{i=1}^{r} \ell^{*}_{AB'} y_{*i}\right)^2}{24r}.$$

The variance AB_{pc} in this case is obtained under the assumption that y_{ij}'s are independent and each has variance σ^2 as

$$\begin{aligned} Var(AB_{pc}) &= \left(\frac{1}{12r}\right)^2 Var\left(\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}\right) \\ &= \left(\frac{1}{12r}\right)^2 Var\left(\left(\sum_{i=1}^{r} \ell^{*'}_{AB2} y_{*i}\right)_{rep2} + \left(\sum_{i=1}^{r} \ell^{*'}_{AB3} y_{*i}\right)_{rep3} + \left(\sum_{i=1}^{r} \ell^{*'}_{AB4} y_{*i}\right)_{rep4}\right) \\ &= \left(\frac{1}{12r}\right)^2 (8r\sigma^2 + 8r\sigma^2 + 8r\sigma^2) \\ &= \frac{\sigma^2}{6r}. \end{aligned}$$

The confounded effects AC is obtained as the average of estimates of AC obtained from the replicates 1, 3 and 4 as

$$AC_{pc} = \frac{AC_{rep1} + AC_{rep3} + AC_{rep4}}{3}$$

$$= \frac{\sum_{i=1}^{r} \ell_{AC}^{*'} y_{*i}}{12r}$$

where (24×1) the vector

$$\ell_{AC'}^{*} = (\ell_{AB1},\ \ell_{AC3},\ \ell_{AC4})$$

consists of 24 elements,

The sum of squares due to AC in this case is given by

$$SS_{AC_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell_{AC}^{*} y_{*i}\right)^2}{24r}.$$

The variance of AC in this case under the assumption that y_{ij} 's are independent and each has variance σ^2 is given by

$$Var(AC_{pc}) = \frac{\sigma^2}{6r}.$$

Similarly, the confounded effect BC is estimated as the average of the estimates of BC obtained from the replicates $1, 2$ and 4 as

$$BC_{pc} = \frac{BC_{rep1} + BC_{rep2} + BC_{rep4}}{3}$$

$$= \frac{\sum_{i=1}^{r} \ell_{BC}^{*'} y_{*i}}{12r}$$

where the vector

$$\ell_{BC}^{*} = (\ell_{BC1},\ \ell_{BC3},\ \ell_{BC4})$$

consists of 24 elements.

The sum of squares due to BC in this case is based on $24r$ elements and is given as

$$SS_{BC_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell_{BC}^{*'} y_{*i}\right)^2}{24r}.$$

The variance of BC in this case is obtained under the assumption that y_{ij}'s are independent and each has variance σ^2 as

$$Var(BC_{pc}) = \frac{\sigma^2}{6r}.$$

Lastly, the confounded effect ABC can be estimated first from the replicates 1,2 and 3 and then the estimate of ABC is obtained as an average of these three individual estimates as

$$ABC_{pc} = \frac{ABC_{rep1} + ABC_{rep2} + ABC_{rep3}}{3}$$

where (24×1) the vector

$$\ell^{*'}_{ABC} = \left(\ell_{ABC1},\ \ell_{ABC2},\ \ell_{ABC3}\right)$$

consists of 24 elements.

The sum of squares due to ABC in this case is based on $24r$ elements and is given by

$$SS_{ABC_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{ABC} y_{*i}\right)^2}{24r}.$$

The variance of ABC in this case assuming that y_{ij}'s are independent and each has variance σ^2 is given by

$$Var(ABC_{pc}) = \frac{\sigma^2}{6r}.$$

If an unconfounded design with $4r$ replication was used then the variance of each of the factors A, B, C, AB, BC, AC and ABC is $\sigma^{*2}/8r$ where σ^{*2} is the error variance on blocks of size 8. So the relative efficiency of a confounded effect in the partially confounded design with respect to that of an unconfounded one in a comparable design is

$$\frac{6r/\sigma^2}{8r/\sigma^{*2}} = \frac{3}{4}\frac{\sigma^{*2}}{\sigma^2}.$$

So the information on a partially confounded effect relative to an unconfounded effect is $\frac{3}{4}$. If $\sigma^{*2} > 4\sigma^2/3,$ then partially confounded design gives more information than the unconfounded design.

Further, the sum of squares due to block can be divided into two components – within replicates and due to replications. So we can write

$$SS_{\text{Block}} = SS_{\text{Block(wr)}} + SS_{\text{Block(r)}}$$

where the sum of squares due to blocks within replications (wr) is

$$SS_{\text{Block(wr)}} = \frac{1}{2^3}\sum_{i=1}^{4r}\left(\frac{B_{1i}^2 + B_{2i}^2}{2} - R_i^2\right)$$

which carries $4r$ degrees of freedom and the sum of squares due to replication is

$$SS_{\text{Block(r)}} = \frac{1}{2^3}\sum_{i=1}^{4r} R_i^2 - \frac{G^2}{32r}$$

which carries $(4r-1)$ degrees of freedom. The total sum of squares is

$$SS_{\text{Total(pc)}} = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{G^2}{32r}.$$

The analysis of variance table in this case of 2^3 factorial under partial confounding is given as

Source	Sum of squares	Degrees of freedom	Mean squares
Replicates	$SS_{Block(r)}$	$4r$ - 1	$MS_{Block(r)}$
Blocks within replicate	$SS_{Block(wr)}$	$4r$	$MS_{Block(wr)}$
Factor A	SS_A	1	MS_A
Factor B	SS_B	1	MS_B
Factor C	SS_C	1	MS_C
AB	$SS_{AB(pc)}$	1	$MS_{AB(pc)}$
AC	$SS_{AC(pc)}$	1	$MS_{AC(pc)}$
BC	$SS_{BC(pc)}$	1	$MS_{BC(pc)}$
ABC	$SS_{ABC(pc)}$	1	$MS_{ABC(pc)}$
Error	by subtraction	$24r$ - 7	$MS_{E(pc)}$
Total	$SS_{Total(pc)}$	$32r$ – 1	

Test of hypothesis can be carried out in the usual way as in the case of factorial experiment.