Chapter 11

Fractional Replications

Consider the set up of complete factorial experiment, say 2^k. If there are four factors, then the total number of plots needed to conduct the experiment is $2^4 = 16$. When the number of factors increases to six, then the required number of plots to conduct the experiment becomes $2^6 = 64$ and so on. Moreover, the number of treatment combinations also become large when the number of factors increases. Sometimes, it is so large that it becomes practically difficult to organize such a huge experiment. Also, the quantity of experimental material needed, time, manpower etc. also increase and sometimes even it may not be possible to have so many resources to conduct a complete factorial experiment. The non-experimental type of errors also enters in the planning and conduct of the experiment. For example, there can be a slip in numbering the treatments or plots or they may be wrongly reported if they are too large in numbers.

About the degree of freedoms, in the 2^6 factorial experiment there are $2^6 - 1 = 63$ degrees of freedom which are divided as 6 for main effects, 15 for two-factor interactions and rest 42 for three or higher-order interactions. In case, the higher-order interactions are not of much use or much importance, then they can possibly be ignored. The information on main and lower-order interaction effects can then be obtained by conducting a fraction of complete factorial experiments. Such experiments are called as **fractional factorial experiments.** The utility of such experiments becomes more when the experimental process is more influenced and governed by the main and lower-order interaction effects rather than the higher-order interaction effects. The fractional factorial experiments need less number of plots and lesser experimental material than required in the complete factorial experiments. Hence it involves less cost, less manpower, less time etc.

It is possible to combine the runs of two or more fractional factorials to assemble sequentially a larger experiment to estimate the factor and interaction effects of interest.

To explain the fractional factorial experiment and its related concepts, we consider here examples in the set up of 2^k factorial experiments.

One-half fraction of 2^3 factorial experiment

First, we consider the set up of 2^3 factorial experiment and consider its one-half fraction. This is a very simple set up to understand the basics, definitions, terminologies and concepts related to the fractional factorials.

Consider the setup of 2^3 factorial experiment consisting of three factors, each at two levels. There is a total of 8 treatment combinations involved. So 8 plots are needed to run the complete factorial experiment.

Suppose the material needed to conduct the complete factorial experiment in 8 plots is not available or the cost of total experimental material is too high. The experimenter has material or money which is sufficient only for four plots. So the experimenter decides to have only four runs, i.e., ½ fraction of 2^3 factorial experiment. Such an experiment contains a one-half fraction of a 2^3 experiment and is called 2^{3-1} factorial experiment. Similarly, $1/2^2$ fraction of 2^3 factorial experiment requires only 2 runs and contains $1/2^2$ fraction of 2^3 factorial experiment and is called as 2^{3-2} factorial experiment. In general, $1/2^p$ fraction of a 2^k factorial experiment requires only 2^{k-p} runs and is denoted as 2^{k-p} factorial experiment.

We consider the case of ½ fraction of 2^3 factorial experiment to describe the various issues involved and to develop the concepts. The first question is how to choose four out of eight treatment combinations for conductive the experiment. In order to decide this, first we have to choose an interaction factor which the experimenter feels can be ignored. Generally, this can be a higher-order interaction which is usually difficult to interpret. We choose ABC in this case. Now we create the table of treatment combinations as in the following table.

Arrangement of treatment combinations for one-half fraction of 2^3 factorial experiment

Factors Treatment combinations	I	A	B	C	AB	AC	BC	ABC
a	+	+	-	-	-	-	+	+
b	+	-	+	-	-	+	-	+
c	+	-	-	+	+	-	-	+
abc	+	+	+	+	+	+	+	+
***********	*******	*******	*******	*******	*******	*******	*******	******
ab	+	+	+	-	+	-	-	-
ac	+	+	-	+	-	+	-	-
bc	+	+	+	+	-	-	+	-
(1)	+	+	-	-	+	+	+	-

This table is obtained by the following steps.

- Write down the factor to be ignored which is ABC in our case. We can express ABC as
  $ABC = (a+b+c+abc)-(ab+ac+bc+(1))$.
- Collect the treatment combinations with plus (+) and minus (–) signs together; divide the eight treatment combinations into two groups with respect to the + and – signs. This is done in the last column corresponding to ABC.
- Write down the symbols + or – of the other factors A, B, C, AB, AC and BC corresponding to (a, b, c, abc) and $(ab, ac, bc, (1))$.

This provides the arrangement of treatments as given in the table. Now consider the column of ABC. The treatment combinations corresponding to + signs of treatment combinations in ABC provide one-half fraction of 2^3 factorial experiment. The remaining treatment combinations corresponding to the – signs in ABC will constitute another one-half fractions of 2^3 factorial experiment. Here one of the one-half fractions corresponding to + signs contains the treatment combinations a, b, c and abc. Another one-half fraction corresponding to - signs contains the treatment combinations ab, ac, bc and (1). Both the one-half fractions are separated by a starred line in the Table.

Generator:

The factor which is used to generate the one-half fractions is called as the **generator.** For example, ABC is the generator of a fraction in the present case and we have two one-half fractions.

Defining relation:

The defining relation for a fractional factorial is the set of all columns that are equal to the identity column I. The identity column I always contains all the + signs. So in our case, $I = ABC$ is called the **defining relation** of this fractional factorial experiment.

The number of degrees of freedom associated with a one-half fraction of 2^3 factorial experiment, i.e., 2^{3-1} factorial experiment is 3 which is essentially used to estimate the main effects.

Now consider the one-half fraction containing the treatment combinations a, b, c and abc (corresponding to + signs in the column of ABC).

The factors A, B, C, AB, AC and BC are now estimated from this block as follows:

$$A = a - b - c + abc,$$
$$B = -a + b - c + abc,$$
$$C = -a - b + c + abc,$$
$$AB = -a - b + c + abc,$$
$$AC = -a + b - c + abc,$$
$$BC = a - b - c + abc.$$

Aliases:

We notice that the estimates of A and BC are same. So it is not possible to differentiate between whether A is being estimated or BC is being estimated. As such, $A = BC$. Similarly, the estimates of B and of AC as well as the estimates of C and of AB are also the same. We write this as $B = AC$ and $C = AB$. So it is not possible to differentiate between B and AC as well as between C and AB in the sense that which one is being estimated. Two or more effects that have this property are called **aliases**. Thus

- A and BC are aliases,
- B and AC are aliases and
- C and AB are aliases.

Note that the estimates of A, B, C, AB, BC, AC and ABC are obtained are one-half fraction set up. These estimates can also be obtained from the complete factorial set up. A question arises that how the estimate of an effect in the two different setups are related? The answer is as follows:

In fact, when we estimate A, B and C in 2^{3-1} factorial experiment, then we are essentially estimating $A + BC, B + AC$ and $C + AB$, respectively in a complete 2^3 factorial experiment. To understand this, consider the setup of a complete 2^3 factorial experiment in which A and BC are estimated by

$$A = -(1) + a - b + ab - c + ac - bc + abc,$$
$$BC = (1) + a - b - ab - c - ac + bc + abc.$$

Adding A and BC and ignoring the common multiplier, we have

$$A + BC = a - b - c + abc$$

which is the same as A or BC is a one-half fraction with $I = ABC$.

Similarly, considering the estimates of B and AC in 2^3 factorial experiment, adding them together and ignoring the common multiplier, we have

$$B = -(1) - a + b + ab - c - ac + bc + abc,$$
$$AC = (1) - a + b - ab + ac - bc + abc,$$
$$B + AC = -a + b - c + abc,$$

which is same B or AC in one-half fraction with $I = ABC$

The estimates of C and AB in 2^3 factorial experiment and their sum is as follows:

$$C = -(1) - a - b - ab + c + ac + bc + abc,$$
$$AB = (1) - a - b + ab + c - ac - bc + abc,$$
$$C + AB = -a - b - c + abc,$$

which is same as C or AB in one-half fraction with $I = ABC$.

Determination of alias structure:

The alias structure is determined by using the defining relation. Multiplying any column (or effect) by the defining relation yields the aliases for that column (or effect). For example, in this case, the defining relation is $I = ABC$. Now multiply the factors on both sides of $I = ABC$ yields

$$A \times I = (A) \times (ABC) = A^2BC = BC,$$
$$B \times I = (B) \times (ABC) = AB^2C = AC,$$
$$C \times I = (C) \times (ABC) = ABC^2 = AB.$$

The systematic rule to find aliases is to write down all the effects of a $2^{3-1} = 2^2$ factorial in a standard order and multiply each factor by the defining contrast.

Alternate or complementary one-half fraction:

We have considered up to now the one-half fraction corresponding to + signs of treatment combinations in ABC column in the table. Now suppose we choose other one-half fraction, i.e., treatment combinations with – signs in ABC column in the table. This is called an **alternate or complementary one-half fraction.** In this case, the effects are estimated as

$$A = ab + ac - bc - (1),$$
$$B = ab - ac + bc - (1),$$
$$C = -ab + ac + bc - (1),$$
$$AB = ab - ac - bc + (1),$$
$$AC = -ab + ac - bc + (1),$$
$$BC = -ab - ac + bc + (1),$$

In this case, we notice that $A=-BC,\ B=-AC,\ C=-AB,$ so the same factors remain aliases again which are aliases in the one-half fraction with + sign in ABC. If we consider the setup of the complete 2^3 factorial experiment, then in case of complete fractional

$$A=-(1)+a-b+ab-c+ac-bc+abc$$
$$BC=(1)+a-b-ab-c-ac+bc+abc,$$

we observe that $A-BC$ in the complete 2^3 factorial experiment the same as A or BC in the one-half fractional with $I=-ABC$ (ignoring the common multiplier). In order to find the relationship between the estimates under this one-half fraction and a complete factorial, we find that what we estimate in the one-half fraction with – sign in ABC is the same as of estimating $A-BC$ from a complete 2^3 factorial experiment. Similarly, using $B-AC$ in the complete 2^3 factorial, is the same as using B or AC is a one-half fraction with $I=ABC$. Using C - AB in a complete 2^3 factorial experiment is the same using as C or AB in one-half fraction with $I=ABC$ (ignoring the common multiplier).

Now there are two one-half fractions corresponding to + and – signs of treatment combinations in ABC. Based on that, there are now two sets of treatment combinations. A question arises that which one to use?

In practice, it does not matter which fraction is actually used. Both the one-half fractions belong to the same family of 2^3 factorial experiment. Moreover, the difference of negative signs in aliases of both the halves becomes positive while obtaining the sum of squares in the analysis of variance.

Use of more than one defining relations:

Further, suppose we want to have $1/2^2$ fraction of 2^3 factorial experiment with one more defining relation, say $I=BC$ along with $I=ABC$. So the one-half with + signs of ABC can further be divided into two halves. In this case, each half fraction will contain two treatments corresponding to

- \+ sign of BC, (viz., a and abc) and
- \- sign of BC, (viz., b and c).

These two halves will constitute the one-fourth fraction of 2^3 factorial experiment. Similarly, we can consider the other one-half fraction corresponding to – sign of ABC. Now we look for + and – sign corresponding to $I=BC$ which constitute the two one-half fractions consisting of the treatments

- (1), bc and
- ab, ac.

This will again constitute the one-fourth fraction of 2^3 factorial experiment.

Example in 2^6 factorial experiment:

In order to have more understanding of the fractional factorial, we consider the setup of 2^6 factorial experiment. Since the highest order interaction, in this case, is $ABCDEF$, so we construct the one-half fraction using $I = ABCDEF$ as defining relation. Then we write all the factors of $2^{6-1} = 2^5$ factorial experiment in the standard order. Then multiply all the factors with the defining relation.

For example,

$$\begin{aligned} I \times A &= ABCDEF \times A \\ &= A^2BCDEF \\ \text{or } A &= BCDEF \end{aligned}$$

Similarly,

$$\begin{aligned} I \times ABC &= ABCDEF \times ABC \\ &= A^2B^2CDEF \\ \text{or } ABC &= CDEF \text{ etc.} \end{aligned}$$

All such operations are illustrated in the following table.

One-half fraction of 2^6 factorial experiment using $I = ABCDEF$ as defining relation

$I = ABCDEF$	$D = ABCEF$	$E = ABCDF$	$DE = ABCF$
$A = BCDEF$	$AD = BCEF$	$AE = BCDF$	$ADE = BCF$
$B = ACDEF$	$BD = ACEF$	$BE = ACDF$	$BDE = ACF$
$AB = CDEF$	$ABD = CEF$	$ABE = CDF$	$ABDE = CF$
$C = ABDEF$	$CD = ABEF$	$CE = ABDF$	$CDE = ABF$
$AC = BDEF$	$ACD = BEF$	$ACE = BDF$	$ACDE = BF$
$BC = ADEF$	$BCD = AEF$	$BCE = ADF$	$BCDE = AF$
$ABC = DEF$	$ABCD = EF$	$ABCE = DF$	$ABCDE = F$

In this case, we observe that

- all the main effects have 5 factor interactions as aliases,
- all the 2 factor interactions have 4 factor interactions as aliases and
- all the 3 factor interactions have 3 factor interactions as aliases.

Suppose a completely randomized design is adopted with blocks of size 16. There are 32 treatments and *abcdef* is chosen as the defining contrast for half replicate. Now all the 32 treatments cannot be accommodated. Only 16 treatments can be accommodated. So the treatments are to be divided and allocated into two blocks of size 16 each. This is equivalent to saying that one factorial effect (and its alias) are confounded with blocks. Suppose we decide that the three-factor interactions and their aliases (which are also three factors interactions in this case) are to be used as an error. So choose one of the three-factor interaction, say ABC (and its alias DEF) to be confounded. Now one of the blocks contains all the treatment combinations having an even number of letters a, b or c. These blocks are constructed in the following table.

One-half replicate of 2^6 factorial experiment in the blocks of size 16

Block 1	Block 2
(1)	*ab*
de	*ae*
df	*af*
ef	*bd*
ab	*be*
ac	*bf*
bc	*cd*
abde	*ce*
abdf	*cf*
abef	*adef*
acde	*bdef*
acdf	*cdef*
acef	*abcd*
bcde	*abce*
bcdf	*abcf*
bcef	*abcdef*

There are total 31 degrees of freedom, out of which 6 degrees of freedom are used by the main effects, 15 degrees of freedom are used by the two-factor interactions and 9 degrees of freedom are used by the error (from three-factor interactions). Additionally, one more division of degree of freedom arises in this case which is due to blocks. So the degree of freedom carried by blocks is 1. That is why the error degrees of freedom are 9 (and not 10) because one degree of freedom goes to block.

- Suppose the block size is to be reduced further and we want to have blocks of size 8 in the same setup. This can be achieved by $1/2^2$ fraction of 2^6 factorial experiment. In terms of confounding setup, this is equivalent to saying that the two factorial effects are to be confounded. Suppose we choose ABD (and its alias CEF) in addition to ABC (and its alias DEF). When we confound two effects, then their generalized interaction also gets confounded. So the interaction $ABC \times ABD = A^2B^2CD = CD$ (or $DEF \times CEF = CDE^2F^2 = CD$) and its alias $ABEF$ also get confounded. One may note that a two-factor interaction is getting confounded in this case which is not a good strategy. A good strategy in such cases where an important factor is getting confounded is to choose the least important two-factor interaction. The blocks arising with this plan are described in the following table. These blocks are derived by dividing each block of the earlier table of a one-half replicate of 2^6 factorial experiment in the blocks of size 16 into two halves. These halves contain respectively an odd and even number of the letters c and d.

Block 1	Block 2	Block 3	Block 4
(1)	*de*	*ae*	*ad*
ef	*df*	*af*	*bd*
ab	*ac*	*be*	*ce*
abef	*bc*	*bf*	*cf*
acde	*abde*	*cd*	*abce*
acdf	*abdf*	*abcd*	*abcf*
bcde	*acef*	*cdef*	*adef*
bcdf	*bcef*	*abcdef*	*bdef*

The total degrees of freedom, in this case, is 31 which are divided as follows:

- the blocks carry 3 degrees of freedom,
- the main effect carries 6 degrees of freedom.
- the two-factor interactions carry 14 degrees of freedom and

the error carries 8 degrees of freedom.

The analysis of variance in the case of fractional factorial experiments is conducted in the usual way, as in the case of any factorial experiment. The sums of squares for blocks, main effects and two-factor interactions are computed in the usual way.

Resolution:

The criterion of resolution is used to compare the fractional factorial designs for overall quality of the statistical inferences which can be drawn. It is defined as the length of the shortest word (or order of the lowest-order effect) aliased with "*I*" in the generating relationship.

A fractional factorial design with the greater resolution is considered to be better than a design with lower resolution. An important objective in the designs is to find a fractional factorial design that has the greatest resolution for a given number of runs and numbers of factors. The resolution of design is generally denoted by a subscripted roman numeral. For example, a fractional factorial design constructed by using $"I = ABCD = \pm ABEF (= \pm CDEF)"$ is denoted as 2_{IV}^{6-2} fractional factorial plan. In practice, the designs with resolution III, IV and V are used in practice.

When the design is of **resolution II,** it means that, e.g., "$I = +AB$". It means that in this case "$A = +B$" which indicates that at least some pairs of main effects are aliased.

When the design is of **resolution III**, the generating relation is like e.g., "$I = +ABC$". In this case "$A = +BC =\ldots$" This means that the main effects will not be aliased with each other but some of them will be aliased with two-factor interaction. Thus such design can estimate all main effects if all interactions are absent.

When the design is of **resolution IV,** then the generating relationship is like "$I = +ABCD$". Then the main effects will not be aliased with each other or with two-factor interactions but some will get aliased with three-factor interaction, e.g., "$A = +BCD$". Some pairs of two-factor interactions will also get aliased, e.g., ., "$AB = +CD =\ldots$". So this type of design unbiasedly estimates all the main effects even when two-factor interactions are present.

Similarly, the generating relations like ., "$I = +ABCDE$" are used in resolution V designs. In this case, all main effects can be estimated unbiasedly in the absence of all interactions of the order less them five. The two-factor interactions can be estimated if no effects of higher-order are present. So resolution V design provides a complete estimation of the second-order model.

The designs of resolution II or higher than resolution V are not used in practice. Reason being that resolution II design cannot separate the influence of main effects and design of resolution VI or higher require a large number of units which may not be feasible all the times.