Chapter 3
Experimental Design Models

We consider the models which are used in designing an experiment. The experimental conditions, experimental setup and the objective of the study primarily determine that what type of design is to be used and hence which type of design model can be used for the further statistical analysis to conclude about the decisions. These models are based on one-way classification, two-way classifications (with or without interactions), etc. We discuss them now in detail in a few setups which can be extended further to any order of classification. We discuss them now under the set up of one-way and two-way classifications.

It may be noted that it has already been described how to develop the likelihood ratio tests for the testing the hypothesis of equality of more than two means from normal distributions and now we will concentrate more on deriving the same tests through the least-squaress principle under the setup of the linear regression model. The design matrix is assumed to be not necessarily of full rank and consists of 0's and 1's only.

One way classification:

Let p random samples from p normal populations with the same variances but different means and different sample sizes have been independently drawn.

Let the observations Y_{ij} follow the linear regression model setup and

Y_{ij} denotes the j^{th} observation of dependent variable Y when the effect of i^{th} level of the factor is present.

Then Y_{ij} are independently normally distributed with

$$E(Y_{ij}) = \mu + \alpha_i, i = 1, 2, ..., p, j = 1, 2, ..., n_i$$
$$V(Y_{ij}) = \sigma^2$$

where

$\mu -$ is the general mean effect.

- is fixed.
- gives an idea about the general conditions of the experimental units and treatments.

$\alpha_i -$ is the effect of i^{th} level of the factor.

- can be fixed or random.

Example: Consider a medicine experiment in which there are three different dosages of drugs - 2 mg., 5 mg., 10 mg. which are given to patients for controlling the fever. These are the 3 levels of drugs, and so denote $\alpha_1 = 2$ mg., $\alpha_2 = 5$ mg., $\alpha_3 = 10$ mg. Let Y denotes the time taken by the medicine to reduce the body temperature from high to normal. Suppose two patients have been given 2 mg. of dosage, so Y_{11} and Y_{12} will denote their responses. So we can write that when $\alpha_1 = 2mg$ is given to the two patients, then

$E(Y_{1j}) = \mu + \alpha_1;\ j = 1, 2.$

Similarly, if $\alpha_2 = 5$ mg. and $\alpha_3 = 10$ mg. of dosages are given to 4 and 7 patients respectively then the responses follow the model

$E(Y_{2j}) = \mu + \alpha_2;\ j = 1, 2, 3, 4$

$E(Y_{3j}) = \mu + \alpha_3;\ j = 1, 2, 3, 4, 5, 6, 7.$

Here μ denotes the general mean effect which may be thought as follows: The human body has a tendency to fight against the fever, so the time taken by the medicine to bring down the temperature depends on many factors like body weight, height, general health condition etc. of the patient. So μ denotes the general effect of all these factors which is present in all the observations.

In the terminology of the linear regression model, μ denotes the intercept term which is the value of the response variable when all the independent variables are set to take value zero. In experimental designs, the models with intercept term are more commonly used and so generally we consider these types of models.

Also, we can express

$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij};\ i = 1, 2, ..., p,\ j = 1, 2, ..., n_i$ where ε_{ij} is the random error component in Y_{ij}. It indicates the variations due to uncontrolled causes which can influence the observations. We assume that ε_{ij} 's are identically and independently distributed as $N(0, \sigma^2)$ with $E(\varepsilon_{ij}) = 0,\ Var(\varepsilon_{ij}) = \sigma^2$.

Note that the general linear model considered is

$$E(Y) = X\beta$$

for which Y_{ij} can be written as

$$E(Y_{ij}) = \beta_i .$$

When all the entries in *X* are 0's or 1's, then this model can also be re-expressed in the form of

$$E(Y_{ij}) = \mu + \alpha_i .$$

This gives rise to some more issues.

Consider and rewrite

$$\begin{aligned} E(Y_{ij}) &= \beta_i \\ &= \bar{\beta} + (\beta_i - \bar{\beta}) \\ &= \mu + \alpha_i \end{aligned}$$

where

$$\mu \equiv \bar{\beta} = \frac{1}{p}\sum_{i=1}^{p} \beta_i$$

$$\alpha_i \equiv \beta_i - \bar{\beta}.$$

Now let us see the changes in the structure of the design matrix and the vector of regression coefficients.

The model $E(Y_{ij}) = \beta_i = \mu + \alpha_i$ can now be rewritten as

$$E(Y) = X^*\beta^*$$
$$Cov(Y) = \sigma^2 I$$

where $\beta^* = (\mu,\ \alpha_1,\ \alpha_2, ...,\ \alpha_p)'$ is a $p \times 1$ vector and

$$X^* = \begin{bmatrix} 1 & \\ 1 & X \\ \vdots & \\ 1 & \end{bmatrix}$$

is a $n \times (p+1)$ matrix, and *X* denotes the earlier defined design matrix in which

- first n_1 rows as (1,0,0,…,0),
- second n_2 rows as (0,1,0,…,0)
- …, and
- last n_p rows as (0,0,0,…,1).

We earlier assumed that $rank(X) = p$ but can we also say that $rank(X^*)$ is also p in the present case?

Since the first column of X^* is the vector sum of all its remaining p columns, so

$rank(X^*) = p$.

It is thus apparent that all the linear parametric functions of $\alpha_1, \alpha_2, ..., \alpha_p$ are not estimable. The question now arises is what kind of linear parametric functions are estimable?

Consider any linear estimator

$$L = \sum_{i=1}^{p} \sum_{j=1}^{n_i} a_{ij} Y_{ij}$$

with

$$C_i = \sum_{j=1}^{n_i} a_{ij}$$

Now

$$\begin{aligned} E(L) &= \sum_{i=1}^{p} \sum_{j=1}^{n_i} a_{ij} \; E(Y_{ij}) \\ &= \sum_{i=1}^{p} \sum_{j=1}^{n_i} a_{ij} \; (\mu + \alpha_i) \\ &= \mu \sum_{i=1}^{p} \sum_{j=1}^{n_i} a_{ij} + \sum_{i=1}^{p} \sum_{j=1}^{n_i} a_{ij} \alpha_i \\ &= \mu (\sum_{i=1}^{p} C_i) + \sum_{i=1}^{p} C_i \alpha_i . \end{aligned}$$

Thus $\sum_{i=1}^{p} C_i \; \alpha_i$ is estimable if and only if

$$\sum_{i=1}^{p} C_i = 0,$$

i.e., $\sum_{i=1}^{p} C_i \alpha_i$ is a contrast.

Thus, in general, neither $\sum_{i=1}^{p} \alpha_i$ nor any $\mu, \alpha_1, \alpha_2, ..., \alpha_p$ is estimable. If it is a contrast, then it is estimable.

This effect and outcome can also be seen from the following explanation based on the estimation of parameters $\mu, \alpha_1, \alpha_2, ..., \alpha_p$.

Consider the least-squares estimation $\hat{\mu}, \hat{\alpha}_1, \hat{\alpha}_2, ..., \hat{\alpha}_p$ of $\mu_1, \alpha_1, \alpha_2, ..., \alpha_p$ respectively.

Minimize the sum of squares due to ε_{ij}'s

$$S = \sum_{i=1}^{p}\sum_{j=1}^{n_i} \varepsilon_{ij}^2 = \sum_{i=1}^{p}\sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i)^2$$

to obtain $\hat{\mu}, \hat{\alpha}_1, ..., \hat{\alpha}_p$.

$$(a) \quad \frac{\partial S}{\partial \mu} = 0 \Rightarrow \sum_{i=1}^{p}\sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i) = 0$$

$$(b) \quad \frac{\partial S}{\partial \alpha_i} = 0 \Rightarrow \sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i) = 0, \; i = 1, 2, ..., p.$$

Note that (a) can be obtained from (b) or vice versa. So (a) and (b) are linearly dependent in the sense that there are (p + 1) unknowns and p linearly independent equations. Consequently $\hat{\mu}, \hat{\alpha}_1, ..., \hat{\alpha}_p$ do not have a unique solution. Same applies to the maximum likelihood estimation of $\mu, \alpha_1, ... \alpha_p$..

If a side condition that

$$\sum_{i=1}^{p} n_i \hat{\alpha}_i = 0 \quad \text{or} \quad \sum_{i=1}^{p} n_i \alpha_i = 0$$

is imposed then (a) and (b) have a unique solution as

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{n_i} y_{ij} = \bar{y}_{oo},$$

$$\hat{\alpha}_i = \frac{1}{n_p}\sum_{j=1}^{n_i} y_{ij} - \hat{\mu}$$

$$= \bar{y}_{io} - \bar{y}_{oo}$$

where $n = \sum_{i=1}^{p} n_i$.

In case, all the sample sizes are the same, then the condition $\sum_{i=1}^{p} n_i \hat{\alpha}_i = 0$ or $\sum_{i=1}^{p} n_i \alpha_i = 0$ reduces to

$\sum_{i=1}^{p} \hat{\alpha}_i = 0$ or $\sum_{i=1}^{p} \alpha_i = 0$.

So the model $y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$ needs to be rewritten so that all the parameters can be uniquely estimated. Thus

$$\begin{aligned} Y_{ij} &= \mu + \alpha_i + \varepsilon_{ij} \\ &= (\mu + \bar{\alpha}) + (\alpha_i - \bar{\alpha}) + \varepsilon_{ij} \\ &= \mu^* + \alpha_i^* + \varepsilon_{ij} \end{aligned}$$

where

$$\mu^* = \mu + \bar{\alpha}$$

$$\alpha_i^* = \alpha_i - \bar{\alpha}$$

$$\bar{\alpha} = \frac{1}{p}\sum_{i=1}^{p} \alpha_i$$

and

$$\sum_{i=1}^{p} \alpha_i^* = 0$$

This is a **reparameterized form** of the linear model.

Thus in a linear model when X is not of full rank, then the parameters do not have unique estimates. In such conditions, a restriction $\sum_{i=1}^{p} \alpha_i = 0$ (or equivalently $\sum_{i=1}^{p} n_i \alpha_i = 0$ in case all n_i*'s* are not the same) can be added and then the least squares (or maximum likelihood) estimators obtained are unique.

The model

$$E(Y_{ij}) = \mu * + \alpha_i^*; \; \sum_{i=1}^{p} \alpha_1^* = 0$$

is called a **reparametrization of the original linear model**.

Let us now consider the analysis of variance with an additional constraint. Let

$$\begin{aligned} Y_{ij} &= \beta_i + \varepsilon_{ij}, \; i = 1, 2, ..., p; \; j = 1, 2, ..., n_i \\ &= \bar{\beta} + (\beta_i - \bar{\beta}) + \varepsilon_{ij} \\ &= \mu + \alpha_i + \varepsilon_{ij} \end{aligned}$$

with

$$\mu = \bar{\beta} = \frac{1}{p}\sum_{i=1}^{p} \beta_i, \quad \alpha_i = \beta_i - \bar{\beta},$$

$$\sum_{i=1}^{p} n_i \alpha_i = 0, \quad n = \sum_{i=1}^{p} n_i.$$

and ε_{ij} 's are identically and independently distributed with mean 0 and variance σ^2.

The null hypothesis is

$H_0 : \alpha_1 = \alpha_2 = ... = \alpha_p = 0$

and the alternative hypothesis is

$H_1 :$ atleast one $\alpha_i \neq \alpha_j$ for all $i, j.$

This model is a one-way layout in the sense that the observations y_{ij}'s are assumed to be affected by only one treatment effect α_i. So the null hypothesis is equivalent to testing the equality of p population means or equivalently the equality of p treatment effects.

We use the principle of least squares to estimate the parameters $\mu, \alpha_1, \alpha_2, ...\alpha_p$.

Minimize the error sum of squares

$$E = \sum_{i=1}^{p} \sum_{j=1}^{n_i} \varepsilon_{ij}^2 = \sum_{i=1}^{p} \sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i)^2$$

with respect to $\mu, \alpha_1, \alpha_2, ..., \alpha_p$. The normal equations are obtained as

$$\frac{\partial E}{\partial \mu} = 0 \Rightarrow -2 \sum_{i=1}^{p} \sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i) = 0$$

or

$$n\mu + \sum_{i=1}^{p} n_i \alpha_i = \sum_{i=1}^{p} \sum_{j=1}^{n_i} y_{ij} \qquad (1)$$

$$\frac{\partial E}{\partial \alpha_i} = 0 \Rightarrow -2 \sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i) = 0$$

or

$$n_i \mu + n_i \alpha_i = \sum_{j=1}^{n_i} y_{ij} \qquad (i = 1, 2, ..., p). \qquad (2)$$

Using $\sum_{i=1}^{p} n_i \alpha_i = 0$ in (1) gives

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{p} \sum_{j=1}^{n_i} y_{ij} = \frac{G}{n} = \bar{y}_{oo}$$

where $G = \sum_{i=1}^{p} \sum_{j=1}^{n_i} y_{ij}$ is the grand total of all the observations.

Substituting $\hat{\mu}$ in (2) gives

$$\hat{\alpha}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij} - \hat{\mu}$$
$$= \frac{T_i}{n_i} - \hat{\mu}$$
$$= \bar{y}_{io} - \bar{y}_{oo}$$

where $T_i = \sum_{j=1}^{n_i} y_{ij}$ is the treatment total due to i^{th} effect α_i, i.e., a total of all the observations receiving the i^{th} treatment and $\bar{y}_{io} = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}$.

Now the fitted model is $y_{ij} = \hat{\mu} + \hat{\alpha}_i$ and the error sum of squares after substituting $\hat{\mu}$ and $\hat{\alpha}_i$ in E becomes

$$E = \sum_{i=1}^{p}\sum_{j=1}^{n_i}(y_{ij} - \hat{\mu} - \hat{\alpha}_i)^2$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{n_i}\left[(y_{ij} - \bar{y}_{oo}) - (\bar{y}_{io} - \bar{y}_{oo})\right]^2$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{oo})^2 - \sum_{i=1}^{p}\sum_{j=1}^{n_i}(\bar{y}_{io} - \bar{y}_{oo})^2$$
$$= \left(\sum_{i=1}^{p}\sum_{j=1}^{n_i} y_{ij}^2 - \frac{G^2}{n}\right) - \left(\sum_{i=1}^{p}\frac{T_i^2}{n_i} - \frac{G^2}{n}\right)$$

where the total sum of squares (TSS)

$$TSS = \sum_{i=1}^{p}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{oo})^2$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{n_i} y_{ij}^2 - \frac{G^2}{n},$$

and $\frac{G^2}{n}$ is called the **correction factor** (CF)**.**

To obtain a measure of variation due to treatments, let

$H_0 = \alpha_1 = \alpha_2 = ... = \alpha_p = 0$

be true. Then the model becomes

$Y_{ij} = \mu + \varepsilon_{ij},\ i = 1, 2, ..., p;\ j = 1, 2, ..., n_i$.

Minimizing the error sum of squares

$$E_1 = \sum_{i=1}^{p} \sum_{j=1}^{n_i} (y_{ij} - \mu)^2$$

with respect to μ, the normal equation is obtained as

$$\frac{\partial E_1}{\partial \mu} = 0 \Rightarrow -2\sum_{i=1}^{p} \sum_{j=1}^{n_i} (y_{ij} - \mu) = 0$$

or

$$\hat{\mu} = \frac{G}{n} = \overline{y}_{oo}.$$

Substituting $\hat{\mu}$ *in* E_1, the error sum of squares becomes

$$\begin{aligned} E_1 &= \sum_{i=1}^{p} \sum_{j=1}^{n_i} (y_{ij} - \hat{\mu})^2 \\ &= \sum_{i=1}^{p} \sum_{j=1}^{n_i} (y_{ij} - \overline{y}_{oo})^2 \\ &= \sum_{i=1}^{p} \sum_{j=1}^{n_i} y_{ij}^2 - \frac{G^2}{n}. \end{aligned}$$

Note that

E_1: Contains variation due to treatment and error both

E: Contains variation due to error only

So $E_1 - E$: contain variation due to treatment only.

The **sum of squares due to treatment** $(SSTr)$ is given by

$$SSTr = E_1 - E$$

$$\begin{aligned} SSTr &= \sum_{i=1}^{p} \sum_{j=1}^{n_i} (\overline{y}_{io} - \overline{y}_{oo})^2 \\ &= \sum_{i=1}^{p} \frac{T_i^2}{n_i} - \frac{G^2}{n}. \end{aligned}$$

The following quantity is called the **error sum of squares** or **sum of squares due to error** (SSE)

$$SSE = \sum_{i=1}^{n} \sum_{j=1}^{n_i} (y_{ij} - \overline{y}_{io})^2.$$

These sum of squares forms the basis for the development of tools in the analysis of variance. We can write

$$TSS = SSTr + SSE.$$

The distribution of degrees of freedom among these sum of squares is as follows:

- The total sum of squares is based on n quantities subject to the constraint that $\sum_{i=1}^{p} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{oo}) = 0$ so TSS carries $(n-1)$ degrees of freedom.

- The sum of squares due to the treatments is based on p quantities subject to the constraint $\sum_{i=1}^{p} n_i (\bar{y}_{io} - \bar{y}_{oo}) = 0$ so $SSTr$ has $(p-1)$ degrees of freedom.
- The sum of squares due to errors is based on n quantities subject to p constraints

  $$\sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{io}) = 0, \; i = 1, 2, ..., p$$

  so SSE carries $(n-p)$ degrees of freedom.

Also, note that

$$TSS = SSTr + SSE,$$

the TSS has been divided into two orthogonal components - $SSTr$ and SSE. Moreover, all TSS, $SSTr$ and SSE can be expressed in a quadratic form. Since ε_{ij} are assumed to be identically and independently distributed following $N(0, \sigma^2)$, so y_{ij} are also independently distributed following $N(\mu + \alpha_i, \; \sigma^2)$.

Now using the theorems 7 and 8 with $q_1 = SSTr, \; q_2 = SSE$, we have under H_0,

$$\frac{SSTr}{\sigma^2} \sim \chi^2(p-1)$$

and

$$\frac{SSE}{\sigma^2} \sim \chi^2(n-p).$$

Moreover, $SSTr$ and SSE are independently distributed.

The mean squares is defined as the sum of squares divided by the degrees of freedom. So the mean square due to treatment is

$$MSTr = \frac{SSTr}{p-1}$$

and the mean square due to error is

$$MSE = \frac{SSE}{n-p}.$$

Thus, under H_0,

$$F = \frac{\left(\frac{MSTr}{\sigma^2}\right)}{\left(\frac{MSE}{\sigma^2}\right)} \sim F(p-1,\ n-p).$$

The decision rule is that reject H_0 if

$$F > F_{1-\alpha,\ p-1,\ n-p}$$

at α % level of significance.

If H_0 does not hold true, then

$$\frac{MSTr}{MSE} \sim noncentral\ F(p-1,\ n-p,\ \delta)$$

where $\delta = \sum_{i=1}^{p} \frac{n_i \alpha_i^2}{\sigma^2}$ is the non-centrality parameter.

Note that the test statistic $\frac{MSTr}{MSE}$ can also be obtained from the likelihood ratio test.

If H_0 is rejected, then we go for multiple comparison tests and try to divide the population into several groups having the same effects.

The analysis of variance table is as follows:

Source of variation	Degrees of freedom	Sum of squares	Mean sum of squares	F-value
Treatment	$p-1$	$SSTr$	$MSTr$	$\frac{MSTr}{MSE}$
Error	$n-p$	SSE	MSE	
Total	$n-1$	TSS		

Now we find the expectations of $SSTr$ and SSE.

$$E(SSTr) = E\left[\sum_{i=1}^{p} n_i (\bar{y}_{io} - \bar{y}_{oo})^2\right]$$
$$= E\left[\sum_{i=1}^{p} n_i \left\{(\mu + \alpha_i + \bar{\varepsilon}_{io}) - (\mu + \bar{\varepsilon}_{oo})\right\}^2\right]$$

where

$$\bar{\varepsilon}_{io} = \frac{1}{n_i} \sum_{j=1}^{n_i} \varepsilon_{ij},\ \bar{\varepsilon}_{oo} = \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{n_i} \varepsilon_{ij} \text{ and } \sum_{i=1}^{p} \frac{n_i \alpha_i}{n} = 0.$$

$$E(SSTr) = E\left[\sum_{i=1}^{p} n_i \left\{\alpha_i + (\bar{\varepsilon}_{io} - \bar{\varepsilon}_{oo})\right\}^2\right]$$
$$= \sum_{i=1}^{p} n_i E(\alpha_i^2) + \sum_{i=1}^{p} n_i E(\bar{\varepsilon}_{io} - \bar{\varepsilon}_{oo})^2 + 0.$$

Since

$$E(SSTr) = \sum_{i=1}^{p} n_i \alpha_i^2 + \sigma^2 \sum_{i=1}^{p} n_i \left(\frac{1}{n_i} - \frac{1}{n}\right)$$
$$= \sum_{i=1}^{p} n_i \alpha_i^2 + (p-1)\sigma^2$$

$$\text{or } E\left(\frac{SSTr}{p-1}\right) = \sigma^2 + \frac{\sum_{i=1}^{p} n_i \alpha_i^2}{p-1}$$

$$\text{or } E(MSTr) = \sigma^2 + \frac{\sum_{i=1}^{p} n_i \alpha_i^2}{p-1}.$$

$$E(\bar{\varepsilon}_{io}^2) = Var(\bar{\varepsilon}_{io}) = Var\left(\frac{1}{n_i}\sum_{j=1}^{n_i} \varepsilon_{ij}\right) = \frac{1}{n_i^2} n_i \sigma^2 = \frac{\sigma^2}{n_i}$$

$$E(\bar{\varepsilon}_{oo}^2) = Var(\bar{\varepsilon}_{oo}) = Var\left(\frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{n_i} \varepsilon_{ij}\right) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

$$E(\bar{\varepsilon}_{io}\bar{\varepsilon}_{oo}) = Cov(\bar{\varepsilon}_{io}, \bar{\varepsilon}_{oo})$$
$$= \frac{1}{n_i n} Cov\left(\sum_{j=1}^{n_i} \varepsilon_{ij} \sum_{i=1}^{p}\sum_{j=1}^{n_i} \varepsilon_{ij}\right)$$
$$= \frac{n_i \sigma^2}{n_i n} = \frac{\sigma^2}{n}.$$

Next

$$E(SSE) = E\left[\sum_{i=1}^{p}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{io})^2\right]$$

$$= E\left[\sum_{i=1}^{p}\sum_{j=1}^{n_i}\left\{(\mu+\alpha_i+\varepsilon_{ij})-(\mu+\alpha_i+\bar{\varepsilon}_{io})\right\}^2\right]$$

$$= E\left[\sum_{i=1}^{p}\sum_{j=1}^{n_i}(\varepsilon_{ij}-\bar{\varepsilon}_{io})^2\right]$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{n_i}E(\varepsilon_{ij}^2+\bar{\varepsilon}_{io}^2-2\varepsilon_{ij}\bar{\varepsilon}_{io})$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{n_i}\left(\sigma^2+\frac{\sigma^2}{n_i}-\frac{2\sigma^2}{n_i}\right)$$

$$= \sigma^2\sum_{i=1}^{p}\sum_{j=1}^{n_i}\left(\frac{n_i-1}{n_i}\right)$$

$$= \sigma^2\sum_{i=1}^{p}\frac{n_i(n_i-1)}{n_i}$$

$$= \sigma^2\sum_{i=1}^{p}(n_i-1)$$

$$= (n-p)\sigma^2$$

or $E\left(\frac{SSE}{n-p}\right)=\sigma^2$

or $E(MSE)=\sigma^2.$

Thus *MSE* is an unbiased estimator of σ^2.

Two-way classification under fixed effects model

Suppose the response of an outcome is affected by the two factors – *A* and *B*. For example, suppose *I* varieties of mangoes are grown on *I* different plots of the same size in each of the *J* different locations. All plots are given the same treatment like an equal amount of water, an equal amount of fertilizer etc. So there are two factors in the experiment which affect the yield of mangoes.

- Location (*A*)
- Variety of mangoes (*B*)

Such an experiment is called a **two – factor experiment**. The different locations correspond to the different levels of *A* and the different varieties correspond to the different levels of factor *B*. The observations are collected on the basis of per plot.

The combined effect of the two factors (*A* and *B* in our case) is called the **interaction effect** (of *A* and *B*).

Mathematically, let *a* and *b* be the levels of factors *A* and *B* respectively then a function $f(a,b)$ is called a function of no interaction if and only if there exists functions $g(a)$ and $h(b)$ such that

$f(a,b) = g(a) + h(b)$.

Otherwise, the factors are said to interact.

For a function $f(a,b)$ of no interaction,

$$f(a_1,b) = g(a_1) + h(b)$$
$$f(a_2,b) = g(a_2) + h(b)$$
$$\Rightarrow f(a_1,b) - f(a_2,b) = g(a_1) - g(a_2)$$

and so it is independent of *b*. Such no interaction functions are called **<u>additive functions</u>**.

Now there are two options:

- Only one observation per plot is collected.
- More than one observations per plot are collected.

If there is only one observation per plot then there cannot be any interaction effect among the observations and we assume it to be zero.

If there are more than one observations per plot then the interaction effect among the observations can be considered.

We consider here two cases

1. One observation per plot in which the interaction effect is zero.
2. More than one observations per plot in which the interaction effect is present.

Two-way classification without interaction

Let y_{ij} be the response of observation from i^{th} level of the first factor, say *A* and j^{th} level of the second factor, say *B*. So assume Y_{ij} are independently distributed as

$N(\mu_{ij},\sigma^2)\;\; i=1,\; 2,...,I,\; j=1,2,...,J.$

This can be represented in the form of a linear model as

$$
\begin{aligned}
E(Y_{ij}) &= \mu_{ij} \\
&= \mu_{oo} + (\mu_{io} - \mu_{oo}) + (\mu_{oj} - \mu_{oo}) + (\mu_{ij} - \mu_{io} - \mu_{oj} + \mu_{oo}) \\
&= \mu + \alpha_i + \beta_j + \gamma_{ij}
\end{aligned}
$$

where

$$
\begin{aligned}
\mu &= \mu_{oo} \\
\alpha_i &= \mu_{io} - \mu_{oo} \\
\beta_j &= \mu_{oj} - \mu_{oo} \\
\gamma_{ij} &= \mu_{ij} - \mu_{io} - \mu_{oj} + \mu_{oo}
\end{aligned}
$$

with

$$
\begin{aligned}
\sum_{i=1}^{I} \alpha_i &= \sum_{i=1}^{I} (\mu_{io} - \mu_{oo}) = 0 \\
\sum_{j=1}^{J} \beta_j &= \sum_{j=1}^{J} (\mu_{oj} - \mu_{oo}) = 0
\end{aligned}
$$

Here

α_i : effect of i^{th} level of factor *A*

or excess of mean of i^{th} level of *A* over the general mean.

β_j : effect of j^{th} level of *B*

or excess of mean of j^{th} level of *B* over the general mean.

γ_{ij} : Interaction effect of i^{th} level of *A* and j^{th} level of *B*.

Here we assume $\gamma_{ij} = 0$ as we have only one observation per plot.

We also assume that the model $E(Y_{ij}) = \mu_{ij}$ is a full rank model so that μ_{ij} and all linear parametric functions of μ_{ij} are estimable.

The total number of observations are $I \times J$ which can be arranged in a two-way classified $I \times J$ table where the rows correspond to the different levels of *A* and the column corresponds to the different levels of *B*.

The observations on Y and the design matrix X in this case are

Y	μ	α_1	α_2	$\cdots$	α_I	β_1	β_2	$\cdots$	β_j
y_{11}	1	1	0	$\cdots$	0	1	0	$\cdots$	0
y_{12}	1	1	0	$\cdots$	0	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots\ddots$		$\vdots$	$\vdots$			
y_{1J}	1	1	0	$\cdots$	0	0	0	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots\ddots$		$\vdots$	$\vdots$	$\vdots\ddots$		$\vdots$
y_{I1}	1	0	0	$\cdots$	1	1	0		0
y_{I2}	1	0	0		1	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots\ddots$		$\vdots$
y_{IJ}	1	0	0		1	0	0		1

If the design matrix is not of full rank, then the model can be reparameterized. In such a case, we can start the analysis by assuming that the model $E(Y_{ij}) = \mu + \alpha_i + \beta_j$ is obtained after reparameterization.

There are two null hypotheses of interest:

$H_{0\alpha} : \alpha_1 = \alpha_2 = ... = \alpha_I = 0$

$H_{0\beta} : \beta_1 = \beta_2 = ... = \beta_J = 0$

against

$H_{1\alpha}$: at least one $\alpha_i (i = 1, 2, ..., I)$ is different from others

$H_{1\beta}$: at least one $\beta_j (j = 1, 2, ..., J)$ is different from others.

Now we derive the least-squares estimators (or equivalently the maximum likelihood estimator) of μ, α_i and β_j, $i = 1, 2, ..., I$, $j = 1, 2, ..., J$ by minimizing the error sum of squares

$$E = \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \mu - \alpha_i - \beta_j)^2.$$

The normal equations are obtained as

$$\frac{\partial E}{\partial \mu} = 0 \Rightarrow -2 \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \mu - \alpha_i - \beta_j) = 0$$

$$\frac{\partial E}{\partial \alpha_i} = 0 \Rightarrow -2 \sum_{j=1}^{J} (y_{ij} - \mu - \alpha_i - \beta_j) = 0 \ , \ i = 1, 2, ..., I \ ,$$

$$\frac{\partial E}{\partial \beta_j} = 0 \Rightarrow -2 \sum_{i=1}^{I} (y_{ij} - \mu - \alpha_i - \beta_j) = 0 \ , \ j = 1, 2, ..., J.$$

Solving the normal equations and using $\sum_{i=1}^{I} \alpha_i = 0$ and $\sum_{j=1}^{J} \beta_j = 0$, the least-squares estimators are obtained as

$$\hat{\mu} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} y_{ij} = \frac{G}{IJ} = \bar{y}_{oo}$$

$$\hat{\alpha}_i = \frac{1}{J} \sum_{j=1}^{J} y_{ij} - \bar{y}_{oo} = \frac{T_i}{J} - \bar{y}_{oo} = \bar{y}_{io} - \bar{y}_{oo} \quad i = 1, 2, ..., I$$

$$\hat{\beta}_j = \frac{1}{I} \sum_{i=1}^{I} y_{ij} - \bar{y}_{oo} = \frac{B_j}{I} - \bar{y}_{oo} = \bar{y}_{oj} - \bar{y}_{oo}, \ j = 1, 2, ..., J$$

where

T_i : treatment totals due to i^{th} α effect, i.e., the sum of all the observations receiving the i^{th} treatment.

B_j : block totals due to j^{th} β effect, i.e., sum of all the observations in the j^{th} block.

Thus the error sum of squares is

$$\begin{aligned} SSE &= \underset{\mu, \alpha_i, \beta_j}{Min}\ E \\ &= \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \hat{\mu}_i - \hat{\alpha}_i - \hat{\beta}_j)^2 \\ &= \sum_{i=1}^{I} \sum_{j=1}^{J} \left[(y_{ij} - \bar{y}_{oo}) - (\bar{y}_{io} - \bar{y}_{oo}) - (\bar{y}_{oj} - \bar{y}_{oo}) \right]^2 \\ &= \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \bar{y}_{io} - \bar{y}_{oj} + \bar{y}_{oo})^2 \\ &= \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \bar{y}_{oo})^2 - J \sum_{i=1}^{I} (\bar{y}_{io} - \bar{y}_{oo})^2 - I \sum_{j=1}^{J} (\bar{y}_{oj} - \bar{y}_{oo})^2 \end{aligned}$$

which carries

$IJ - (I-1) - (J-1) - 1 = (I-1)(J-1)$ degrees of freedom.

Next, we consider the estimation of μ and β_j under the null hypothesis $H_{0\alpha} : \alpha_1 = \alpha_2 = ... = \alpha_I = 0$ by minimizing the error sum of squares

$$E_1 = \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \mu - \beta_j)^2 .$$

The normal equations are obtained by

$$\frac{\partial E_1}{\partial \mu} = 0 \text{ and } \frac{\partial E_1}{\partial \beta_j} = 0, \ j = 1, 2, ..., J$$

which on solving gives the least square estimates

$$\hat{\mu} = \bar{y}_{oo}$$

$$\hat{\beta}_j = \bar{y}_{oj} - \bar{y}_{oo}.$$

The sum of squares due to $H_{0\alpha}$ is

$$\underset{\mu,\beta_j}{Min}\, E_1 = \underset{\mu,\beta_j}{Min}\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\mu-\beta_j)^2$$
$$=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\hat{\mu}-\hat{\beta}_j)^2$$
$$= J\sum_{i=1}^{I}(\overline{y}_{io}-\overline{y}_{oo})^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\overline{y}_{io}-\overline{y}_{oj}+\overline{y}_{oo})^2.$$

↓ ↓

Sum of squares due to factor *A* | Error sum of squares

Thus the sum of squares due to deviation from $H_{0\alpha}$ (or sum of squares due to rows or sum of squares are to factor *A*)

$$SSA = J\sum_{i=1}^{I}(\overline{y}_{io}-\overline{y}_{oo})^2 = J\sum_{i=1}^{I}\overline{y}_{io}^2 - IJ\overline{y}_{oo}^2$$

and carries $(IJ-J)-(I-1)(J-1)=I-1$ degrees of freedom.

Now we find the estimates of μ and α_i under $H_{0\beta}:\beta_1=\beta_2=...=\beta_J=0$ by minimizing

$$E_2=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\mu-\alpha_i)^2.$$

The normal equations are

$$\frac{\partial E_2}{\partial \mu}=0 \text{ and } \frac{\partial E_2}{\partial \alpha_i}=0,\; i=1,2,...,I$$

which on solving give the estimators as

$$\hat{\mu}=\overline{y}_{oo}$$
$$\hat{\alpha}_i=\overline{y}_{io}-\overline{y}_{oo}.$$

The minimum value of the error sum of squares is obtained by

$$\underset{\mu,\alpha_j}{Min}\; E_2=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\hat{\mu}-\hat{\alpha}_i)^2$$
$$=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\overline{y}_{io})^2$$
$$=I\sum_{j=1}^{J}(\overline{y}_{oj}-\overline{y}_{oo})^2+\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\overline{y}_{io}-\overline{y}_{oj}+\overline{y}_{oo})^2$$

↓ ↓

Sum of squares due to factor *B* | Error sum of squares

The sum of squares due to deviation from $H_{0\beta}$ (or the sum of squares due to columns or sum of squares due to factor *B*) is

$$SSB = I\sum_{j=1}^{J}(\bar{y}_{oj} - \bar{y}_{oo})^2 = I\sum_{j}\bar{y}_{oj}^2 - IJ\ \bar{y}_{oo}^2$$

and its degrees of freedom are

$$(IJ - I) - (I-1)(J-1) = J-1.$$

Note that the total sum of squares is

$$\begin{aligned}
TSS &= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{oo})^2 \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J}\left[(\bar{y}_{io} - \bar{y}_{oo}) + (\bar{y}_{oj} - \bar{y}_{oo}) + (y_{ij} - \bar{y}_{io} - \bar{y}_{oj} + \bar{y}_{oo})\right]^2 \\
&= J\sum_{i=1}^{I}(\bar{y}_{io} - \bar{y}_{oo})^2 + I\sum_{j=1}^{J}(\bar{y}_{oj} - \bar{y}_{oo})^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{io} - \bar{y}_{oj} + \bar{y}_{oo})^2 \\
&= SSA + SSB + SSE.
\end{aligned}$$

The partitioning of degrees of freedom into the corresponding groups is

$$IJ - 1 = (I-1) + (J-1) + (I-1)(J-1).$$

Note that SSA, SSB and SSE are mutually orthogonal and that is why the degrees of freedom can be divided like this.

Now using the theory explained while discussing the likelihood ratio test or assuming y_{ij}'s to be independently distributed as $N(\mu + \alpha_i + \beta_j, \sigma^2),\ i = 1,2,...,I;\ j = 1,2,...,J,$ and using the Theorems 6 and 7, we can write

$$\frac{SSA}{\sigma^2} \sim \chi^2(I-1),\ \ \frac{SSB}{\sigma^2} \sim \chi^2(J-1),\ \ \text{and}\ \ \frac{SSE}{\sigma^2} \sim \chi^2((I-1)(J-1)).$$

So the test statistic for $H_{0\alpha}$ is obtained as

$$\begin{aligned}
F_1 &= \frac{\left(\dfrac{SSA/\sigma^2}{I-1}\right)}{\left(\dfrac{SSE\ /\sigma^2}{(I-1)(J-1)}\right)} \\
&= \frac{(I-1)(J-1)}{(I-1)} \cdot \frac{SSA}{SSE} \\
&= \frac{MSA}{MSE} \sim F((I-1),\ (I-1)\ (J-1)) \text{ under } H_{0\alpha}
\end{aligned}$$

where

$$MSA = \frac{SSA}{I-1}$$

$$MSE = \frac{SSE}{(I-1)(J-1)}.$$

The same statistic is also obtained using the likelihood ratio test for $H_{0\alpha}$.

The decision rule is

Reject $H_{0\alpha}$ if $F_1 > F_{1-\alpha}\left[(I-1),(I-1)\ (J-1)\right]$.

Under $H_{1\alpha}$, F_1 follows a noncentral F distribution $F(\delta,(J-1),(I-1)(J-1))$ where $\delta = \frac{J\sum_{i=1}^{I}\alpha_i^2}{\sigma^2}$ is the associated non-centrality parameter.

Similarly, the test statistic for $H_{0\beta}$ is obtained as

$$F_2 = \frac{\left(\frac{SSB/\sigma^2}{J-1}\right)}{\left(\frac{SSE/\sigma^2}{(I-1)(J-1)}\right)} = \frac{(I-1)(J-1)}{(J-1)}\frac{SSB}{SSE} = \frac{MSB}{MSE} \sim F((J-1),(I-1)(J-1)) \text{ under } H_{0\beta}$$

where $MSB = \frac{SSB}{J-1}$.

The decision rule is

Reject $H_{0\beta}$ if $F_2 > F_{1-\alpha}((J-1),(I-1)(J-1))$.

The same test statistic can also be obtained from the likelihood ratio test.

The analysis of variance table is as follows:

Source of variation	Degrees of freedom	Sum of squares	Mean sum of squares	F-value
Factor A (or rows)	$(I-1)$	SSA	MSA	$F_1 = \frac{MSA}{MSE}$
Factor B (or column)	$(J-1)$	SSB	MSB	$F_2 = \frac{MSB}{MSE}$
Error	$(I-1)(J-1)$	SSE (by subtraction)	MSE	
Total	$IJ-1$	TSS		

It can be found on similar lines as in the case of one way classification that

$$E(MSA) = \sigma^2 + \frac{J}{I-1}\sum_{i=1}^{I}\alpha_i^2$$

$$E(MSB) = \sigma^2 + \frac{I}{J-1}\sum_{j=1}^{J}\beta_j^2$$

$$E(MSE) = \sigma^2.$$

If the null hypothesis is rejected, then we use the multiple comparison tests to divide the α_i's (or β_j's) into groups such that α_i's (or β_j's) belonging to the same group are equal and those belonging to different groups are different. Generally, in practice, the interest of experimenter is more in using the multiple comparison test for treatment effects rather on the block effects. So the multiple comparison test are used generally for the treatment effects only.

Two-way classification with interactions:

Consider the two-way classification with an equal number, say K observations per cell. Let

y_{ijk} : k^{th} observation in $(i, j)^{th}$ cell , i.e., receiving the treatments i^{th} level of factor A and j^{th} level of factor B, $i = 1, 2, ..., I;\ j = 1, 2, ..., J; k = 1, 2, ..., K$ and

y_{ijk} are independently drawn from $N(\mu_{ij}, \sigma^2)$ so that the linear model under consideration is

$$y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$$

where ε_{ijk} are identically and independently distributed following $N(0, \sigma^2)$. Thus

$$\begin{aligned} E(y_{ijk}) &= \mu_{ij} \\ &= \mu_{oo} + (\mu_{io} - \mu_{oo}) + (\mu_{oj} - \mu_{oo}) + (\mu_{ij} - \mu_{io} - \mu_{oj} + \mu_{oo}) \\ &= \mu + \alpha_i + \beta_j + \gamma_{ij} \end{aligned}$$

where

$$\mu = \mu_{oo}$$
$$\alpha_i = \mu_{io} - \mu_{oo}$$
$$\beta_j = \mu_{oj} - \mu_{oo}$$
$$\gamma_{ij} = \mu_{ij} - \mu_{io} - \mu_{oj} + \mu_{oo}$$

with

$$\sum_{i=1}^{I}\alpha_i = 0, \sum_{j=1}^{J}\beta_j = 0, \sum_{i=1}^{I}\gamma_{ij} = 0, \sum_{j=1}^{J}\gamma_{ij} = 0.$$

Assume that the design matrix X is of full rank so that all the parametric functions of μ_{ij} are estimable.

The null and the corresponding alternative hypotheses are

$H_{0\alpha}: \alpha_1 = \alpha_2 = ... = \alpha_I = 0$
$H_{1\alpha}:$ At least one $\alpha_i \neq \alpha_j$, for $i \neq j$,

$H_{0\beta}: \beta_1 = \beta_2 = ... = \beta_J = 0$
$H_{1\beta}:$ At least one $\beta_i \neq \beta_j$, for $i \neq j$

and

$H_{0\gamma}:$ All $\gamma_{ij} = 0$ for all i, j.
$H_{1\gamma}:$ At least one $\gamma_{ij} \neq \gamma_{ik}$, for $j \neq k$.

Minimizing the error sum of squares

$$E = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2,$$

The normal equations are obtained as

$$\frac{\partial E}{\partial \mu} = 0, \ \frac{\partial E}{\partial \alpha_i} = 0 \ \text{ for all } i, \ \frac{\partial E}{\partial \beta_j} = 0 \ \text{ for all } j \ \text{ and } \ \frac{\partial E}{\partial \gamma_{ij}} = 0 \ \text{ for all } i \text{ and } j$$

The least-squares estimates are obtained as

$$\hat{\mu} = \bar{y}_{ooo} = \frac{1}{IJK} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} y_{ijk}$$

$$\hat{\alpha}_i = \bar{y}_{ioo} - \bar{y}_{ooo} = \frac{1}{JK} \sum_{i=1}^{I} y_{ijk} - \bar{y}_{ooo}$$

$$\hat{\beta}_j = \bar{y}_{ojo} - \bar{y}_{ooo} = \frac{1}{IK} \sum_{j=1}^{J} y_{ijk} - \bar{y}_{ooo}$$

$$\hat{\gamma}_{ij} = \bar{y}_{ijo} - \bar{y}_{ioo} - \bar{y}_{ojo} + \bar{y}_{ooo} = \frac{1}{K} \sum_{i=1}^{I} \sum_{j=1}^{J} y_{ijk} - \bar{y}_{ioo} - \bar{y}_{ojo} + \bar{y}_{ooo}.$$

The error sum of squares is

$$\begin{aligned} SSE &= \underset{\hat{\mu}, \hat{\alpha}_i, \hat{\beta}_j, \hat{\gamma}_{ij}}{Min} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2 \\ &= \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\gamma}_{ij})^2 \\ &= \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \bar{y}_{ijo})^2 \end{aligned}$$

with $\frac{SSE}{\sigma^2} \sim \chi^2(IJ(K-1))$.

Now minimizing the error sum of squares under $H_{0\alpha} = \alpha_1 = \alpha_2 = ... = \alpha_I = 0$, i.e., minimizing

$E_1 = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \mu - \beta_j - \gamma_{ij})^2$ with respect to μ, β_j and γ_{ij} and solving the normal equations

$$\frac{\partial E_1}{\partial \mu} = 0, \ \frac{\partial E_1}{\partial \beta_j} = 0 \text{ for all } j \text{ and } \frac{\partial E_1}{\partial \gamma_{ij}} = 0 \text{ for all } i \text{ and } j$$

gives the least squares estimates as

$$\hat{\mu} = \hat{y}_{ooo}$$
$$\hat{\beta}_j = \bar{y}_{ojo} - \bar{y}_{ooo}$$
$$\hat{\gamma}_{ij} = \bar{y}_{ijo} - \bar{y}_{ooo} - \bar{y}_{ojo} + \bar{y}_{ooo} = \bar{y}_{ijo} - \bar{y}_{ojo}.$$

The sum of squares due to $H_{0\alpha}$, is

$$\begin{aligned}
\underset{\mu,\beta_j,\gamma_{ij}}{Min} \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \mu - \beta_j - \gamma_{ij})^2
&= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \hat{\mu} - \hat{\beta}_j - \hat{\gamma}_{ij})^2 \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ijo})^2 + JK\sum_{i=1}^{I}(\bar{y}_{ioo} - \bar{y}_{ooo})^2 \\
&= SSE + JK \sum_{i=1}^{I}(\bar{y}_{ioo} - \bar{y}_{ooo})^2.
\end{aligned}$$

Thus the sum of squares due to deviation from $H_{0\alpha}$ or the sum of squares due to effect A is

$$SSA = \text{Sum of squares due to } H_{0\alpha} - \text{SSE} = JK\sum_{i=1}^{I}(\bar{y}_{ioo} - \bar{y}_{ooo})^2$$

with $\frac{SSA}{\sigma^2} \sim \chi^2(I-1).$

Minimizing the error sum of squares under $H_{0\beta} : \beta_1 = \beta_2 = ... = \beta_J = 0$, i.e., minimizing

$$E_2 == \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \mu - \alpha_i - \gamma_{ij})^2 ,$$

and solving the normal equations

$$\frac{\partial E_2}{\partial \mu} = 0, \ \frac{\partial E_2}{\partial \alpha_i} = 0 \text{ for all } j \text{ and } \frac{\partial E_2}{\partial \gamma_{ij}} = 0 \text{ for all } i \text{ and } j$$

yields the least-squares estimators as

$$\hat{\mu} = \bar{y}_{ooo}$$
$$\hat{\alpha}_i = \bar{y}_{ioo} - \bar{y}_{ooo}$$
$$\hat{\gamma}_{ij} = \bar{y}_{ijo} - \bar{y}_{ooo} - \bar{y}_{ioo} + \bar{y}_{ooo} = \bar{y}_{ijo} - \bar{y}_{ioo}.$$

The minimum error sum of squares is

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-\hat{\mu}-\hat{\alpha}_i-\hat{\gamma}_{ij})^2$$
$$= SSE + IK\sum_{j=1}^{J}(\overline{y}_{ojo}-\overline{y}_{ooo})^2$$

and the sum of squares due to deviation from $H_{o\beta}$ or the sum of squares due to effect B is

$$SSB = \text{ Sum of squares due to } H_{0\beta} - SSE = IK\sum_{j=1}^{J}(\overline{y}_{ojo}-\overline{y}_{ooo})^2$$

with $\dfrac{SSB}{\sigma^2} \sim \chi^2(J-1).$

Next, minimizing the error sum of squares under $H_{0\gamma}: all\ \gamma_{ij}=0$ for all *i, j*, i.e., minimizing

$$E_3 = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-\mu-\alpha_i-\beta_j)^2$$

with respect to μ, α_i and β_j and solving the normal equations

$$\frac{\partial E_3}{\partial \mu}=0,\ \frac{\partial E_3}{\partial \alpha_i}=0 \text{ for all } i \text{ and } \frac{\partial E_3}{\partial \beta_j}=0 \text{ for all } j$$

yields the least-squares estimators

$$\hat{\mu}=\overline{y}_{ooo}$$
$$\hat{\alpha}_i=\overline{y}_{ioo}-\overline{y}_{ooo}$$
$$\hat{\beta}_j=\overline{y}_{ojo}-\overline{y}_{ooo}.$$

The sum of squares due to $H_{0\gamma}$ is

$$\underset{\mu,\alpha_i,\beta_j}{Min}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-\mu-\alpha_i-\beta_j)^2$$
$$=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-\hat{\mu}-\hat{\alpha}_i-\hat{\beta}_j)^2$$
$$= SSE + K\sum_{i=1}^{I}\sum_{j=1}^{J}(\overline{y}_{ijo}-\overline{y}_{ioo}-\overline{y}_{ojo}+\overline{y}_{ooo})^2.$$

Thus the sum of squares due to deviation from $H_{0\gamma}$ or the sum of squares due to the interaction effect AB is

$$SSAB = \text{ Sum of squares due to } H_{0\gamma} - SSE = K\sum_{i=1}^{I}\sum_{j=1}^{J}(\overline{y}_{ijo}-\overline{y}_{ioo}-\overline{y}_{ojo}+\overline{y}_{ooo})^2$$

with $\dfrac{SSAB}{\sigma^2} \sim \chi^2((I-1)J-1)).$

The total sum of squares can be partitioned as

$$TSS = SSA + SSB + SSAB + SSE$$

where SSA, SSB, $SSAB$ and SSE are mutually orthogonal. So either using the independence of SSA, SSB, $SSAB$ and SSE as well as their respective χ^2-distributions or using the likelihood ratio test approach, the decision rules for the null hypothesis at α level of significance are based on *F*-statistic as follows

$$F_1 = \frac{IJ(K-1)}{I-1}.\frac{SSA}{SSE} \sim F\left[(I-1, IJ(K-1)\right] \text{ under } H_{0\alpha},$$

$$F_2 = \frac{IJ(K-1)}{J-1}.\frac{SSB}{SSE} \sim F\left[(J-1, IJ(K-1)\right] \text{ under } H_{0\beta},$$

and

$$F_3 = \frac{IJ(K-1)}{(I-1)(J-1)}.\frac{SSAB}{SSE} \sim F\left[(I-1)(J-1), IJ(K-1)\right] \text{ under } H_{0\gamma}.$$

So

Reject $H_{0\alpha}$ if $F_1 > F_{1-\alpha}\left[(I-1), IJ(K-1)\right]$

Reject $H_{0\beta}$ if $F_2 > F_{1-\alpha}\left[(J-1), IJ(K-1)\right]$

Reject $H_{0\gamma}$ if $F_3 > F_{1-\alpha}\left[(I-1)(J-1), IJ(K-1)\right]$.

If $H_{0\alpha}$ or $H_{0\beta}$ is rejected, one can use *t*-test or multiple comparison test to find which pairs of α_i's or β_j's are significantly different.

If $H_{0\gamma}$ is rejected, one would not usually explore it further but theoretically, *t*-test or multiple comparison tests can be used.

It can also be shown that

$$E(MSA) = \sigma^2 + \frac{JK}{I-1}\sum_{i=1}^{I}\alpha_i^2$$

$$E(MSB) = \sigma^2 + \frac{IK}{J-1}\sum_{j=1}^{J}\beta_j^2$$

$$E(MSAB) = \sigma^2 + \frac{K}{(I-1)(J-1)}\sum_{i=1}^{I}\sum_{j=1}^{J}\gamma_{ij}^2$$

$$E(MSE) = \sigma^2.$$

The analysis of variance table is as follows:

Source of variation	Degrees of freedom	Sum of squares	Mean sum of squares	*F*-value
Factor *A*	$(I-1)$	SSA	$MSA = \frac{SSA}{I-1}$	$F_1 = \frac{MSA}{MSE}$
Factor *B*	$(J-1)$	SSB	$MSB = \frac{SSB}{J-1}$	$F_2 = \frac{MSB}{MSE}$
Interaction *AB*	$(I-1)(J-1)$	$SSAB$	$MSAB = \frac{SSAB}{(I-1)(J-1)}$	$F_3 = \frac{MSAB}{MSE}$
Error	$IJ(K-1)$	SSE	$MSE = \frac{SSE}{IJ(K-1)}$	
Total	$(IJK-1)$	TSS		

Tukey's test for nonadditivity:

Consider the set up of two way classification with one observation per cell and interaction as

$$y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ij}, \quad i = 1,2...,I,\ j = 1,2,...,J \text{ with } \sum_{i=1}^{I} \alpha_i = 0, \sum_{j=1}^{J} \beta_j = 0.$$

The distribution of degrees of freedom in this case is as follows:

Source	**Degrees of freedom**
A	$I-1$
B	$J-1$
AB(interaction)	$(I-1)(J-1)$
Error	0
Total	$IJ-1$

There is no degree of freedom for error. The problem is that the two factor interaction effect and random error component are subsumed together and cannot be separated out. There is no estimate for σ^2.

If no interaction exists, then $H_0 : \gamma_{ij} = 0$ for all *i* ,*j* is accepted and the additive model

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

is well enough to test the hypothesis $H_0 : \alpha_i = 0$ and $H_0 : \beta_j = 0$ with error having $(I-1)(J-1)$ degrees of freedom.

If interaction exists, then $H_0 : \gamma_{ij} = 0$ is rejected. In such a case, if we assume that the structure of the interaction effect is such that it is proportional to the product of individual effects, i.e.,

$$\gamma_{ij} = \lambda \alpha_i \beta_j$$

then a test for testing $H_0 : \lambda = 0$ can be constructed. Such a test will serve as a test for nonadditivity. It will help in knowing the effect of the presence of interact effect and whether the interaction enters into the model additively. Such a test is given by Tukey's test for nonadditivity which requires one degree of freedom leaving $(I-1)(J-1)-1$ degrees of freedom for error.

Let us assume that departure from additivity can be specified by introducing a product term and writing the model as

$$E(y_{ij}) = \mu + \alpha_i + \beta_j + \lambda \alpha_i \beta_j ;\; i = 1, 2, ..., I,\; j = 1, 2, ..., J \text{ with } \sum_{i=1}^{I} \alpha_i = 0,\; \sum_{j=1}^{J} \beta_j = 0 .$$

When $\lambda \neq 0,$ the model becomes a nonlinear model and the least-squares theory for linear models is not applicable.

Note that using $\sum_{i=1}^{I} \alpha_i = 0,\; \sum_{j=1}^{J} \beta_j = 0$, we have

$$\begin{aligned}
\bar{y}_{oo} &= \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} y_{ij} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \left[\mu + \alpha_i + \beta_j + \lambda \alpha_i \beta_j + \varepsilon_{ij} \right] \\
&= \mu + \frac{1}{I} \sum_{i=1}^{I} \alpha_i + \frac{1}{J} \sum_{j=1}^{J} \beta_j + \frac{\lambda}{IJ} (\sum_{i=1}^{I} \alpha_j)(\sum_{j=1}^{J} \beta_j) + \bar{\varepsilon}_{oo} \\
&= \mu + \bar{\varepsilon}_{oo}
\end{aligned}$$

$$E(\bar{y}_{oo}) = \mu$$
$$\Rightarrow \hat{\mu} = \bar{y}_{oo}.$$

Next

$$\bar{y}_{io} = \frac{1}{J}\sum_{j=1}^{J} y_{ij} = \frac{1}{J}\sum_{j=1}^{J}\left[\mu + \alpha_i + \beta_j + \lambda\alpha_i\beta_j + \varepsilon_{ij}\right]$$

$$= \mu + \alpha_i + \frac{1}{J}\sum_{j=1}^{J}\beta_j + \lambda\alpha_i \frac{1}{J}\sum_{j=1}^{J}\beta_j + \bar{\varepsilon}_{io}$$

$$= \mu + \alpha_i + \bar{\varepsilon}_{io}$$

$$E(\bar{y}_{io}) = \mu + \alpha_i$$

$$\Rightarrow \hat{\alpha}_i = \bar{y}_{io} - \hat{\mu} = \bar{y}_{io} - \bar{y}_{oo}.$$

Similarly

$$\bar{y}_{oj} = \mu + \beta_j$$

$$\Rightarrow \hat{\beta}_j = \bar{y}_{oj} - \hat{\mu} = \bar{y}_{oj} - \bar{y}_{oo}$$

Thus $\hat{\mu}, \hat{\alpha}_i$ and $\hat{\beta}_j$ remain the unbiased estimators of μ, α_i and β_j, respectively irrespective of whether $\lambda = 0$ or not.

Also

$$E\left[y_{ij} - \bar{y}_{io} - \bar{y}_{oj} + \bar{y}_{oo}\right] = \lambda\alpha_i\beta_j$$

or

$$E\left[(y_{ij} - \bar{y}_{oo}) - (\bar{y}_{io} - \bar{y}_{oo}) - (\bar{y}_{oj} - \bar{y}_{oo})\right] = \lambda\alpha_i\beta_j.$$

Consider the estimation of μ, α_i, β_j and λ based on the minimization of

$$S = \sum_i \sum_j (y_{ij} - \mu - \alpha_i - \beta_j - \lambda\alpha_i\beta_j)^2$$

$$= \sum_i \sum_j S_{ij}^2.$$

The normal equations are solved as

$$\frac{\partial S}{\partial \mu} = 0 \Rightarrow \sum_{i=1}^{I}\sum_{j=1}^{J} S_{ij} = 0$$

$$\Rightarrow \hat{\mu} = \bar{y}_{oo}$$

$$\frac{\partial S}{\partial \alpha_i} = 0 \Rightarrow \sum_{j=1}^{J}(1+\lambda\beta_j)S_{ij} = 0$$

$$\frac{\partial S}{\partial \beta_j} = 0 \Rightarrow \sum_{i=1}^{I}(1+\lambda\alpha_i)S_{ij} = 0$$

$$\frac{\partial S}{\partial \lambda} = 0 \Rightarrow \sum_{i=1}^{I}\sum_{j=1}^{J}\alpha_i\beta_j S_{ij} = 0$$

$$\text{or} \sum_{i=1}^{I}\sum_{j=1}^{J}\alpha_i\beta_j(y_{ij}-\mu-\alpha_i-\beta_j-\lambda\alpha_i\beta_j) = 0$$

$$\text{or } \lambda = \frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\alpha_i\beta_j y_{ij}}{\left(\sum_{i=1}^{I}\alpha_i^2\right)\left(\sum_{j=1}^{J}\beta_j^2\right)} = \tilde{\tilde{\lambda}} \text{ (say)}$$

which can be estimated provided α_i and β_j are assumed to be known.

Since α_i and β_j can be estimated by $\hat{\alpha}_i = \bar{y}_{io} - \bar{y}_{oo}$ and $\hat{\beta}_j = \bar{y}_{oj} - \bar{y}_{oo}$ irrespective of whether $\lambda = 0$ or not, so we can substitute them in place of α_i and β_j in $\tilde{\lambda}$ which gives

$$\hat{\lambda} = \frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\hat{\alpha}_i\hat{\beta}_j y_{ij}}{\left(\sum_{i=1}^{I}\hat{\alpha}_i^2\right)\left(\sum_{j=1}^{J}\hat{\beta}_j^2\right)} = \frac{(IJ)\sum_{i=1}^{I}\sum_{j=1}^{J}\hat{\alpha}_i\hat{\beta}_j y_{ij}}{\left(J\sum_{i=1}^{I}\hat{\alpha}_i^2\right)\left(I\sum_{j=1}^{J}\hat{\beta}_j^2\right)}$$

$$= \frac{IJ\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{io}-\bar{y}_{oo})(\bar{y}_{oj}-\bar{y}_{oo})y_{ij}}{S_A S_B}$$

where $S_A = J\sum_{i=1}^{I}\hat{\alpha}_i^2 = J\sum_{i=1}^{I}(\bar{y}_{io}-\bar{y}_{oo})^2$

$$S_B = I\sum_{j=1}^{J}\hat{\beta}_j^2 = I\sum_{j=1}^{J}(\bar{y}_{oj}-\bar{y}_{oo})^2.$$

Assuming α_i and β_j to be known

$$Var(\tilde{\lambda}) = \left(\frac{1}{\sum_{i=1}^{I} \alpha_i^2 \sum_{j=1}^{J} \beta_j^2} \right)^2 \left[\sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_i^2 \beta_j^2 Var(y_{ij}) + 0 \right]$$

$$= \frac{\sigma^2 \left(\sum_{i=1}^{I} \alpha_i^2 \right) \left(\sum_{j=1}^{J} \beta_j^2 \right)}{\left(\sum_{i=1}^{I} \alpha_i^2 \right)^2 \left(\sum_{j=1}^{J} \beta_j^2 \right)^2}$$

$$= \frac{\sigma^2}{\left(\sum_{i=1}^{I} \alpha_i^2 \right) \left(\sum_{j=1}^{J} \beta_j^2 \right)}$$

using $Var(y_{ij}) = \sigma^2,\ Cov(y_{ij}, y_{jk}) = 0$ for all $i \neq k.$

When α_i and β_j are estimated by $\hat{\alpha}_i$ and $\hat{\beta}_j$, then substitute them back in the expression of $Var(\tilde{\lambda})$ and treating it as $Var(\hat{\lambda})$ gives

$$Var(\hat{\lambda}) = \frac{\sigma^2}{\left(\sum_{i=1}^{I} \hat{\alpha}_i^2 \right) \left(\sum_{j=1}^{J} \hat{\beta}_j^2 \right)}$$

$$= \frac{IJ\sigma^2}{S_A S_B}$$

for given $\hat{\alpha}_i$ and $\hat{\beta}_j$.

Note that if $\lambda = 0,$ then

$$E\left[\hat{\lambda} / \hat{\alpha}_i, \hat{\beta}_j \text{ for all } i, j \right] = E \left[\frac{\sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_i \beta_j y_{ij}}{\sum_{i=1}^{I} \alpha_i^2 \sum_{j=1}^{J} \beta_j^2} \right]$$

$$= E \left[\frac{\sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_i \beta_j (\mu + \alpha_i + \beta_j + 0 + \varepsilon_{ij})}{(\sum_{i=1}^{I} \alpha_i^2)(\sum_{j=1}^{J} \beta_j^2)} \right]$$

$$= \frac{0}{(\sum_{i=1}^{I} \alpha_i^2)\ (\sum_{j=1}^{J} \beta_j^2)} = 0.$$

As $\hat{\alpha}_i$ and $\hat{\beta}_j$ remains valid irrespective of $\lambda = 0$ or not, in this sense $\hat{\lambda}$ is a function of y_{ij} and hence normally distributed as

$$\hat{\lambda} \sim N\left(0, \frac{IJ\sigma^2}{S_A S_B}\right).$$

Thus the statistic

$$\frac{(\hat{\lambda})^2}{Var(\hat{\lambda})} = \frac{IJ\left[\sum_{i=1}^{I}\sum_{j=1}^{J}\hat{\alpha}_i\hat{\beta}_j y_{ij}\right]^2}{\sigma^2 S_A S_B}$$

$$= \frac{IJ\left[\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{io}-\bar{y}_{oo})(\bar{y}_{oj}-\bar{y}_{oo})y_{ij}\right]^2}{\sigma^2 S_A S_B}$$

$$= \frac{IJ\left[\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{io}-\bar{y}_{oo})(\bar{y}_{oj}-\bar{y}_{oo})(y_{ij}-\bar{y}_{io}-\bar{y}_{oj}+\bar{y}_{oo})\right]^2}{\sigma^2 S_A S_B}$$

$$= \frac{S_N}{\sigma^2}$$

follows a χ^2 - distribution with one degree of freedom where

$$S_N = \frac{IJ\left[\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{io}-\bar{y}_{oo})(\bar{y}_{oj}-\bar{y}_{oo})(y_{ij}-\bar{y}_{io}-\bar{y}_{oj}+\bar{y}_{oo})\right]^2}{S_A S_B}$$

is the sum of squares due to nonadditivity.

Note that

$$\frac{S_{AB}}{\sigma^2} = \frac{\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{io}-\bar{y}_{oj}+\bar{y}_{oo})^2}{\sigma^2}$$

follows $\chi^2((I-1)(J-1))$.

so $\left(\frac{S_N}{\sigma^2} - \frac{SAB}{\sigma^2}\right)$ is nonnegative and follows $\chi^2\left[(I-1)(J-1)-1\right]$.

The reason for this is as follows:

$y_{ij} = \mu + \alpha_i + \beta_j + \text{non additivity} + \varepsilon_{ij}$

and so

$TSS = SSA + SSB + S_N + SSE$

$\Rightarrow SSE = TSS - SSA - SSB - S_N$

has degrees of freedom

$= (IJ-1) - (I-1) - (J-1) - 1$

$= (I-1)(J-1) - 1$

We need to ensure that $SSE > 0$. So using the result

"If Q, Q_1 and Q_2 are quadratic forms such that

$Q = Q_1 + Q_2$ with $Q \sim \chi^2(a)$, $Q_2 \sim \chi^2(b)$ and Q_2 is non-negative, then

$Q_1 \sim \chi^2(a-b)$"

ensures that the difference

$$\frac{S_N}{\sigma^2} - \frac{SAB}{\sigma^2}$$

is nonnegative.

Moreover S_N (*SS* due to nonadditivity) and *SSE* are orthogonal. Thus the *F*-test for nonadditivity is

$$F = \frac{\left(\dfrac{S_N / \sigma^2}{1}\right)}{\left(\dfrac{SSE/\sigma^2}{(I-1)(J-1)-1}\right)}$$

$$= \left[(I-1)(J-1)-1\right]\frac{SSN}{SSE}$$

$\sim F\left[1, (I-1)(J-1)-1\right]$ under H_0.

So the decision rule is

Reject $H_0 : \lambda = 0$ whenever

$$F > F_{1-\alpha}\left[1, (I-1)(J-1)-1\right]$$

The analysis of variance table for the model including a term for nonadditivity is as follows:

Source of variation	Degrees of freedom	Sum of squares	Mean sum of squares	F-value
A	$I-1$	S_A	$MS_A = \frac{S_A}{I-1}$	
B	$J-1$	S_B	$MS_B = \frac{S_B}{J-1}$	
Nonadditivity	1	S_N	$MS_N = S_N$	$\frac{MS_N}{MSE}$
Error	$(I-1)(J-1)-1$	SSE (By subtraction)	$MSE = \frac{SSE}{(I-1)(J-1)-1}$	
Total	$IJ-1$	TSS		

Comparison of Variances

One of the basic assumptions in the analysis of variance is that the samples are drawn from different normal populations with different means but the same variances. So before going for analysis of variance, the test of hypothesis about the equality of variance is needed to be done.

We discuss the test of equality of two variances and more than two variances.

Case 1: Equality of two variances

$H_0 : \sigma_1^2 = \sigma_2^2 = \sigma^2.$

Suppose there are two independent random samples

$A : x_1, x_2, ..., x_{n_1};\ x_i \sim N(\mu_A, \sigma_A^2)$

$B : y_1, y_2, ..., y_{n_2};\ y_i \sim N(\mu_B, \sigma_B^2)$

The sample variance corresponding to the two samples are

$$s_x^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}(x_i - \bar{x})^2$$

$$s_y^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}(y_i - \bar{y})^2.$$

Under $H_0 : \sigma_A^2 = \sigma_B^2 = \sigma^2$,

$$\frac{(n_1-1)s_x^2}{\sigma^2} \sim \chi^2(n_1-1)$$

$$\frac{(n_2-1)s_y^2}{\sigma^2} \sim \chi^2(n_2-1).$$

Moreover, the sample variances s_x^2 and s_y^2 and are independent. So

$$\left(\frac{\dfrac{\left(\dfrac{(n_1-1)s_x^2}{\sigma^2}\right)}{n_1-1}}{\dfrac{\left(\dfrac{(n_2-1)s_y^2}{\sigma^2}\right)}{n_2-1}} \right) = \frac{s_x^2}{s_y^2} \sim F_{n_1-1,\, n_2-1}.$$

So for testing $H_0 : \sigma_1^2 = \sigma_2^2$ versus $H_1 : \sigma_1^2 \neq \sigma_2^2$, the null hypothesis H_0 is rejected if

$$F > F_{1-\frac{\alpha}{2};n_1-1,n_2-1} \text{ or } F < F_{1-\frac{\alpha}{2};n_1-1,n_2-1} \text{ where } F_{\frac{\alpha}{2};n_1-1,n_2-1;} = \frac{1}{F_{1-\frac{\alpha}{2};n_2-1,n_1-1}}.$$

If the null hypothesis $H_0 : \sigma_A^2 = \sigma_B^2$ is rejected, then the problem is termed as the Fisher-Behren's problem. The solutions are available for this problem.

Case 2: Equality of more than two variances: Bartlett's test

$H_0 : \sigma_1^2 = \sigma_2^2 = ... = \sigma_k^2$ and $H_1 : \sigma_i^2 \neq \sigma_j^2$ for atleast one $i \neq j = 1, 2, ..., k.$

Let there be k independent normal population $N(\mu_i, \sigma_i^2)$ each of size $n_i, i = 1, 2, ..., k.$ Let $s_1^2, s_2^2, ..., s_k^2$ be k independent unbiased estimators of population variances $\sigma_1^2, \sigma_2^2, ..., \sigma_k^2$ respectively with $\nu_1, \nu_2, ..., \nu_k$ degrees of freedom. Under H_0, all the variances are the same as σ^2, say and an unbiased estimate of σ^2 is $s^2 = \sum_{i=1}^{k} \frac{\nu_i s_i^2}{\nu}$ where $\nu_i = n_i - 1, \nu = \sum_{i=1}^{k} \nu_i.$

Bartlett has shown that under H_0

$$\frac{\sum_{i=1}^{k} \left(\nu_i \ \ln \frac{s^2}{s_i^2} \right)}{\left[1 + \frac{1}{3(k-1)} \left\{ \sum_{i=1}^{k} \left(\frac{1}{\nu_i} \right) - \frac{1}{\nu} \right\} \right]}$$

is distributed as $\chi^2(k-1)$ based on which H_0 can be tested.