Chapter 5
Incomplete Block Designs

If the number of treatments to be compared is large, then we need a large number of blocks to accommodate all the treatments. This requires more experimental material and so the cost of experimentation becomes high which may be in terms of money, labour, time etc. The completely randomized design and randomized block design may not be suitable in such situations because they will require a large number of experimental units to accommodate all the treatments. In such cases, when the sufficient number of homogeneous experimental units are not available to accommodate all the treatments in a block, then incomplete block designs can be used. In incomplete block designs, each block receives only some of the selected treatments and not all the treatments. Sometimes it is possible that the available blocks can accommodate only a limited number of treatments due to several reasons. For example, the goodness of a car is judged by different features like fuel efficiency, engine performance, body structure etc. Each of this factor depends on many other factors, e.g., the engine consists of many parts and the performance of every part combined together will result in the final performance of the engine. These factors can be treated as treatment effects. If all these factors are to be compared, then we need a large number of cars to design a complete experiment. This may be an expensive affair. The incomplete block designs overcome such problems. It is possible to use much less number of cars with the set up of incomplete block design and all the treatments need not be assigned to all the cars. Rather some treatments will be implemented in some cars and remaining treatments in other cars. The efficiency of such designs is, in general, not less than the efficiency of a complete block design. In another example, consider a situation of destructive experiments, e.g., testing the life of television sets, LCD panels, etc. If there are a large number of treatments to be compared, then we need a large number of television sets or LCD panels. The incomplete block designs can use a lesser number of television sets or LCD panels to conduct the test of the significance of treatment effects without losing, in general, the efficiency of the design of the experiment. This also results in the reduction of experimental cost. Similarly, in any experiment involving animals like as biological experiments, one would always like to sacrifice fewer animals. Moreover, the government guidelines also restrict the experimenter to use a smaller number of animals. In such cases, either the number of treatments to be compared can be reduced depending upon the number of animals in each block or to reduce the block size. In such cases when the number of treatments to be compared is larger than the number of animals in each block, then the block size is reduced and the setup of incomplete block designs can be used. This will result in a lower cost of experimentation. The incomplete block designs need less number of observations in a block than the observations in a complete block design to conduct the test of hypothesis without losing the efficiency of the design of experiment, in general.

Complete and incomplete block designs:

The designs in which every block receives all the treatments are called the complete block designs.

The designs in which every block does not receive all the treatments but only some of the treatments are called incomplete block design.

The block size is smaller than the total number of treatments to be compared in the incomplete block designs.

There are three types of analysis in the incomplete block designs

- intrablock analysis,
- interblock analysis and
- recovery of interblock information.

Intrablock analysis:

In intrablock analysis, the treatment effects are estimated after eliminating the block effects and then the analysis and the test of significance of treatment effects are conducted further. If the blocking factor is not marked, then the intrablock analysis is sufficient enough to provide reliable, correct and valid statistical inferences.

Interblock analysis:

There is a possibility that the blocking factor is important and the block totals may carry some important information about the treatment effects. In such situations, one would like to utilize the information on block effects (instead of removing it as in the intrablock analysis) in estimating the treatment effects to conduct the analysis of design. This is achieved through the interblock analysis of an incomplete block design by considering the block effects to be random.

Recovery of interblock information:

When both the intrablock and the interblock analysis have been conducted, then two estimates of treatment effects are available from each of the analysis. A natural question then arises -- Is it possible to pool these two estimates together and obtain an improved estimator of the treatment effects to use it for the construction of test statistic for testing of hypothesis? Since such an estimator comprises of more information to estimate the treatment effects, so this is naturally expected to provide better statistical inferences. This is achieved by combining the intrablock and interblock analysis together through the recovery of interblock information.

Intrablock analysis of incomplete block design:

We start here with the usual approach involving the summations over different subscripts of y's. Then gradually, we will switch to a matrix-based approach so that the reader can compare both the approaches. They can also learn the one-to-one relationships between the two approaches for better understanding.

Notations and normal equations:

Let

- v treatments have to be compared.
- b blocks are available.
- k_i : Number of plots in i^{th} block ($i = 1,2,\ldots,b$).
- r_j :Number of plots receiving j^{th} treatment ($j = 1,2,\ldots,v$).
- n: Total number of plots.

  $n = r_1 + r_2 + ... + r_v = k_1 + k_2 + ... + k_b.$
- Each treatment may occur more than once in each block

  or

  may not occur at all.
- n_{ij} denotes the number of times the j^{th} treatment occurs in i^{th} block

For example, $n_{ij} = 1$ or 0 for all i, j means that no treatment occurs more than once in a block and treatment may not occur in some blocks at all. Similarly, $n_{ij} = 1$ means that j^{th} treatment occurs in i^{th} block and $n_{ij} = 0$ means that j^{th} treatment does not occurs in i^{th} block.

It may be noticed that

$$\sum_{j=1}^{v} n_{ij} = k_i \qquad i = 1,...,b$$

$$\sum_{i} n_{ij} = r_j \qquad j = 1,...,v$$

$$n = \sum_{i}\sum_{j} n_{ij}$$

Model:

Let y_{ijm} denotes the response (yield) from the m^{th} replicate of j^{th} treatment in i^{th} block and

$$y_{ijm} = \beta_i + \tau_j + \varepsilon_{ijm} \qquad i = 1,2,...,b,\ j = 1,2,..,v,\ m = 1,2,...,n_{ij}$$

[**Note**: We are not considering here the general mean effect in this model for better understanding of the issues in the estimation of parameters. Later, we will consider it in the analysis.]

Following notations are used in further description.

Block totals : $B_1, B_2, ..., B_b$ where $B_i = \sum_{j}\sum_{m} y_{ijm}$.

Treatment totals: $V_1, V_2, ..., V_v$ where $V_j = \sum_{i}\sum_{m} y_{ijm}$

Grand total : $Y = \sum_{i}\sum_{j}\sum_{m} y_{ijm}$

Generally, a design is denoted by $D(v,b,r,k,n)$ where *v, b, r, k* and *n* are the parameters of the design.

Example:

Let us consider an example to understand the meaning of these notations. Suppose there are 3 blocks (Block 1, Block 2 and Block 3) and 5 treatments (τ_1, τ_2, τ_3, τ_4, τ_5). So b = 3 and v = 5. These treatments are arranged in different plots in blocks as follows:

Block 1: 5 plots

Plot 1 τ_1	Plot 2 τ_1	Plot 3 τ_2	Plot 4 τ_2	Plot 5 τ_3

Block 2: 4 plots

Plot 1 τ_2	Plot 2 τ_4	Plot 3 τ_5	Plot 4 τ_5

Block 3: 2 plots

Plot 1 τ_2	Plot 2 τ_2

k_i

Number of plots in Block 1: $k_1 = 5$

Number of plots in Block 2: $k_2 = 3$

Number of plots in Block 1: $k_3 = 2$

r_j

Number of times τ_1 appears in

- Block 1 = 2
- Block 2 = 0
- Block 3 = 0.

Total number of times τ_1 appears in the entire design is $r_1 = 2 + 0 + 0 = 2$.

Number of times τ_2 appears in

- Block 1 = 2
- Block 2 = 1
- Block 3 = 2.

Total number of times τ_2 appears in the entire design is $r_2 = 2 + 1 + 2 = 5$.

Number of times τ_3 appears in

- Block 1 = 1
- Block 2 = 0
- Block 3 = 0.

Total number of times τ_3 appears in the entire design is $r_3 = 1 + 0 + 0 = 1$.

Number of times τ_4 appears in

- Block 1 = 0
- Block 2 = 1
- Block 3 = 0.

Total number of times τ_4 appears in the entire design is $r_4 = 0 + 1 + 0 = 1$.

Number of times τ_5 appears in

- Block 1 = 0
- Block 2 = 2
- Block 3 = 0.

Total number of times τ_5 appears in the entire design is $r_5 = 0 + 2 + 0 = 2$.

n_{ij}

Total number of times τ_1 appears in Block 1: $n_{11} = 2$

Total number of times τ_2 appears in Block 1: $n_{12} = 2$

Total number of times τ_3 appears in Block 1: $n_{13} = 1$

Total number of times τ_4 appears in Block 1: $n_{14} = 0$

Total number of times τ_5 appears in Block 1: $n_{15} = 0$

Total number of times τ_1 appears in Block 2: $n_{21} = 0$

Total number of times τ_2 appears in Block 2: $n_{22} = 1$

Total number of times τ_3 appears in Block 2: $n_{23} = 0$

Total number of times τ_4 appears in Block 2: $n_{24} = 1$

Total number of times τ_5 appears in Block 2: $n_{25} = 2$

Total number of times τ_1 appears in Block 3: $n_{31} = 0$

Total number of times τ_2 appears in Block 3: $n_{32} = 2$

Total number of times τ_3 appears in Block 3: $n_{33} = 0$

Total number of times τ_4 appears in Block 3: $n_{34} = 0$

Total number of times τ_5 appears in Block 3: $n_{35} = 0$

y_{ijm}

y_{ijm} : response from the m^{th} replicate of j^{th} treatment in i^{th} block, $i = 1,2,3$; $j = 1,2,3,4,5$; $m = 1,2,\ldots, n_{ij}$

Following are the notations for for y_{ijm} different treatments in the blocks

Block 1:	τ_1 y_{111}	τ_1 y_{112}	τ_2 y_{121}	τ_2 y_{122}	τ_3 y_{131}
Block 2:	τ_2 y_{221}	τ_4 y_{241}	τ_5 y_{251}	τ_5 y_{252}	
Block 3:	τ_2 y_{321}	τ_2 y_{322}			

B_i	V_j
$B_1 = y_{111} + y_{112} + y_{121} + y_{122} + y_{131}$ $B_2 = y_{221} + y_{241} + y_{251} + y_{252}$ $B_3 = y_{321} + y_{322}$	$V_1 = y_{111} + y_{112}$ $V_2 = y_{121} + y_{122} + y_{221} + y_{321} + y_{322}$ $V_3 = y_{131}$ $V_4 = y_{241}$ $V_5 = y_{251} + y_{252}$

Normal equations:

Minimizing $S = \sum_i \sum_j \sum_m \varepsilon_{ijm}^2$ with respect to β_i and τ_j, we obtain the least-squares estimators of the parameters as follows:

$$S = \sum_i \sum_j \sum_m (y_{ijm} - \beta_i - \tau_j)^2$$

$$\frac{\partial S}{\partial \beta_i} = 0$$

$$\Rightarrow \sum_j \sum_m (y_{ijm} - \beta_i - \tau_j) = 0$$

or

$$B_i - \beta_i \sum_j \sum_m 1 - \sum_j \tau_j \sum_m 1 = 0 \qquad (1)$$

or

$$B_i = \beta_i k_i + n_{i1}\tau_1 + n_{i2}\tau_2 + ... + n_{iv}\tau_v, \; i = 1,...,b$$

$$B_i = \beta_i k_i + \sum_j \tau_j n_{ij} \qquad [b \text{ equations}]$$

$$\frac{\partial S}{\partial \tau_i} = 0$$

$$\Rightarrow \sum_i \sum_m (y_{ijm} - \beta_i - \tau_j) = 0$$

or $\sum_i \sum_m y_{ijm} - \sum_i \beta_i \sum_m 1 - \tau_j \sum_i \sum_m 1 = 0$

$$V_j - \sum_i \beta_i n_{ij} - \tau_j \sum_i n_{ij} = 0 \qquad (2)$$

or $V_j = n_{ij}\beta_1 + n_{2j}\beta_2 + ... + n_{bj}\beta_b + r_j\tau_j, \; j = 1, 2, ..., v$

or $V_j = \sum_i \beta_i n_{ij} + r_j \tau_j \quad [v \text{ equations}]$

Equations (1) and (2) constitute $(b + v)$ equations.

Note that

$$\sum_i \text{equation (1)} = \sum_j \text{equation (2)}$$

$$\sum_i B_i = \sum_j V_j$$

$$\sum_i \left(\sum_j \sum_m y_{ijm} \right) = \sum_j \left(\sum_i \sum_m y_{ijm} \right).$$

Thus there are at most $(b + v - 1)$ degrees of freedom for estimates. So the estimates of only $(b + v - 1)$ parameters can be obtained out of all $(b + v)$ parameters.

[**Note**: We will see later that degrees of freedom may be less than or equal to $(b + v - 1)$ in special cases. Also, note that we have not assumed any side conditions like $\sum_i \alpha_i = \sum_j \beta_j = 0$ as in the case of complete block designs.]

To obtain the estimates of the parameters, there are two options-

1. Using equation (1), eliminate β_i from equation (2) to estimate τ_j or
2. Using equation (2), eliminate τ_j from equation (1) to estimate β_i.

We consider first the approach 1., i.e., using equation (1), eliminate β_i from equation (2).

From equation (1),

$$\beta_i = \frac{1}{k_i} \left[B_i - \sum_{j=1}^{v} n_{ij} \tau_j \right].$$

Use it in (2) as follows.

$$V_j = n_{ij}\beta_1 + ... + n_{bj}\beta_b + r_j \tau_j$$

$$= n_{1j} \left[\frac{1}{k_1} (B_1 - n_{11}\tau_1 - ... - n_{1v}\tau_{1v}) \right]$$

$$+ n_{2j} \left[\frac{1}{k_2} (B_2 - n_{21}\tau_1 - ... - n_{2v}\tau_v) \right] + ...$$

$$+ n_{bj} \left[\frac{1}{k_b} (B_b - n_{b1}\tau_1 - ... - n_{bv}\tau_v) \right] + r_j \tau_j$$

or

$$V_j - \frac{n_{1j}B_1}{k_1} - \frac{n_{2j}B_2}{k_2} - \ldots - \frac{n_{bj}B_b}{k_b}$$

$$= \tau_1\left[-\frac{n_{11}n_{1j}}{k_1} - \frac{n_{21}n_{2j}}{k_2} - \ldots. - \frac{n_{b1}n_{bj}}{k_b} \right] + \ldots$$

$$+ \tau_v\left[-\frac{n_{1v}n_{1j}}{k_1} - \frac{n_{2v}n_{2j}}{k_2} - \ldots. - \frac{n_{bv}n_{bj}}{k_b} \right] + r_j\tau_j, \quad j = 1,\ldots,v$$

or

$$V_j - \sum_{i=1}^{b} \frac{n_{ij}B_i}{k_i} = \tau_1\left[-\frac{n_{11}n_{1j}}{k_1} \ldots - \frac{n_{b1}n_{bj}}{k_b} \right] + \ldots + \tau_v\left[-\frac{n_{1v}n_{1j}}{k_1} \ldots - \frac{n_{bv}n_{bj}}{k_b} \right] + r_j\tau_j$$

or

$$Q_j = \tau_1\left[-\frac{n_{11}n_{1j}}{k_1} \ldots - \frac{n_{b1}n_{bj}}{k_b} \right] + \ldots + \tau_v\left[-\frac{n_{1v}n_{1j}}{k_1} \ldots - \frac{n_{bv}n_{bj}}{k_b} \right] + r_j\tau_j, \quad j = 1\ldots.v$$

where

$$Q_j = V_j - \left[\frac{n_{1j}B_1}{k_1} + \ldots + \frac{n_{bj}B_b}{k_b} \right], \; j = 1,2,\ldots,v$$

are called **adjusted treatment totals.**

[**Note**: Compared to the earlier case, the j^{th} treatment total V_j is adjusted by a factor $\sum_{i=1}^{b} \frac{n_{ij}B_i}{k_i}$, that is why it is called "adjusted". The adjustment is being made for the block effects because they were eliminated to estimate the treatment effects.]

Note that

k_i : Number of plots in i^{th} block.

$\frac{B_i}{k_i}$ is called the average (response) yield per plot from i^{th} block.

$\frac{n_{ij}B_i}{k_i}$ is considered as an average contribution to the j^{th} treatment total from the i^{th} block.

Q_j is obtained by removing the sum of the average contributions of the b blocks from the j^{th} treatment total V_j.

Write

$$Q_j = \tau_1\left[-\frac{n_{11}n_{1j}}{k_1}...-\frac{n_{b1}n_{bj}}{k_b}\right]+...+\tau_v\left[-\frac{n_{1v}n_{1j}}{k_1}...-\frac{n_{bv}n_{bj}}{k_b}\right]+r_j\tau_j,\;\; j=1,2,...,v.$$

as

$$Q_j = C_{j1}\tau_1 + C_{j2}\tau_2 + ... + C_{jv}\tau_v$$

where

$$C_{jj} = r_j - \frac{n_{1j}^2}{k_1} - \frac{n_{2j}^2}{k_2} - ... - \frac{n_{bj}^2}{k_b}$$

$$C_{jj'} = -\frac{n_{1j}n_{1j'}}{k_1} - \frac{n_{2j}n_{2j'}}{k_2} - ... - \frac{n_{bj}n_{bj'}}{k_b};\;\; j \neq j',\; j=1,2,..,v.$$

The $v \times v$ matrix $C = ((C_{jj'})),\; j=1,2,...,v;\, j'=1,2,...,v$ with C_{jj} as diagonal elements and $C_{jj'}$ as off-diagonal elements is called the ***C*-matrix** of the incomplete block design.

C matrix is symmetric. Its row sum and column sum are zero. (proved later)

Rewrite

$$Q_j = \tau_1\left[-\frac{n_{11}n_{1j}}{k_1}...-\frac{n_{b1}n_{bj}}{k_b}\right]+...+\tau_v\left[-\frac{n_{1v}n_{1j}}{k_1}...-\frac{n_{bv}n_{bj}}{k_b}\right]+r_j\tau_j,\;\; j=1,2,...,v.$$

as

$$Q = C\tau.$$

This equation is called as **reduced normal equations** where $Q' = (Q_1, Q_2, .., Q_v),\; \tau' = (\tau_1, \tau_2, ..., \tau_v)$

Equations (1) and (2) are EQUIVALENT.

Alternative presentation in matrix notations:

Now let us try to represent and translate the same algebra in matrix notations.

Let

$E_{mn} : m \times n$ matrix whose all elements are unity.

$N = (n_{ij})$ is $b \times v$ matrix called as **incidence matrix.**

$$k_i = \sum_{j=1}^{v} n_{ij}$$

$$r_j = \sum_{i=1}^{b} n_{ij}$$

$$n = \sum_i \sum_j n_{ij}$$

$$E_{1b}N = (r_1, r_2, ..., r_v) = r'$$

$$NE_{v1} = (k_1, k_2, ..., k_b)' = k.$$

For illustration, we verify one of the relationships as follows.

$$E_{1b} = (1,1,...,1)_{1\times b}$$

$$N = \begin{pmatrix} n_{11} & n_{12} \cdots n_{1v} \\ n_{21} & n_{22} \cdots n_{2v} \\ \vdots & \vdots \quad \ddots \vdots \\ n_{b1} & n_{b2} \cdots n_{bv} \end{pmatrix}$$

$$\begin{aligned} E_{1b}N &= (1,1,...,1)_{1\times b} \begin{pmatrix} n_{11} & n_{12} \cdots n_{1v} \\ n_{21} & n_{22} \cdots n_{2v} \\ \vdots & \vdots \quad \ddots \vdots \\ n_{b1} & n_{b2} \cdots n_{bv} \end{pmatrix}_{b\times v} \\ &= \left(\sum_{i=1}^{b} n_{i1}, \sum_{i=1}^{b} n_{i2}, ..., \sum_{i=1}^{b} n_{iv} \right) \\ &= (r_1, r_2, ..., r_v) \\ &= r'. \end{aligned}$$

It is now clear that the treatment and blocks are not estimable as such as in the case of complete block designs. Note that we have not made any assumption like $\sum_i \alpha_i = \sum_j \beta_j = 0$ also.

Now we introduce the general mean effect (denoted by μ) in the linear model and carry out further analysis on the same lines as earlier.

Consider the model

$y_{ijm} = \mu + \beta_i + \tau_j + \varepsilon_{ijm}, \quad i = 1,2,...,b; \; j = 1,2,...,v; m = 0,1,..,n_{ij}.$

The normal equations are obtained by minimizing $S = \sum_i \sum_j \sum_m \varepsilon_{ijm}^2$ with respect to the parameters μ, β_i and τ_j and solving them, we can obtain the least-squares estimators of the parameters.

Minimizing $S = \sum_i \sum_j \sum_m \varepsilon_{ijm}^2$ with respect to the parameters μ, β_i and τ_j, the normal equations are obtained as

$$\begin{aligned} & n\hat{\mu} + \sum_i n_{io}\hat{\beta}_i + \sum_j n_{oj}\hat{\tau}_j = G \\ & n_i\hat{\mu} + n_{io}\hat{\beta}_i + \sum_j n_{ij}\hat{\tau}_j = B_i \qquad i = 1,...,b \\ & n_{oj}\hat{\mu} + n_{oj}\hat{\tau}_j + \sum_i n_{ij}\hat{\beta}_i = V_j \qquad j = 1,...,v. \end{aligned}$$

Now we write these normal equations in matrix notations. Denote

$$\beta = Col(\beta_1, \beta_2, ..., \beta_b)$$
$$\tau = Col(\tau_1, \tau_2, ..., \tau_v)$$
$$B = Col(B_1, B_2, ..., B_b) \qquad \backslash$$
$$V = Col(V_1, V_2 ..., V_v)$$
$$N = ((n_{ij})): \text{ incidence matrix of order } b \times v$$

where *Col* (.) denotes the column vector.

Let

$$K = diag(k_1, .., k_b): b \times b \text{ diagonal matrix}$$
$$R = diag(r_1, .., r_v): v \times v \text{ diagonal matrix.}$$

Then the $(b + v + 1)$ normal equations can be written as

$$\begin{pmatrix} G \\ B \\ V \end{pmatrix} = \begin{pmatrix} n & E_{1b}K & E_{1v}R \\ KE_{b1} & K & N \\ RE_{v1} & N' & R \end{pmatrix} \begin{pmatrix} \hat{\mu} \\ \hat{\beta} \\ \hat{\tau} \end{pmatrix}. \qquad (*)$$

Since we are presently interested in the testing of hypothesis related to the treatment effects, so we eliminate the block effects $\hat{\beta}$ to estimate the treatment effects. For doing so, multiply both sides on the left of equation (*) by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & I_b & -NR^{-1} \\ 0 & -N'K^{-1} & I_v \end{bmatrix}$$

where

$$R^{-1} = diag\left(\frac{1}{r_1}, \frac{1}{r_2}, ..., \frac{1}{r_v}\right), \quad K^{-1} = diag\left(\frac{1}{k_1}, \frac{1}{k_2} ..., \frac{1}{k_b}\right).$$

Solving it further, we get set of three equations as follows:

$$G = n\hat{\mu} + E_{1b}K\hat{\beta} + E_{iv}R\hat{\tau}$$
$$B - NR^{-1}V = [K - NR^{-1}N']\hat{\beta}$$
$$V - N'K^{-1}B = [R - N'K^{-1}N]\hat{\tau}$$

These are called as '**reduced normal equations**' or '**reduced intrablock equations**'.

The same reduced intrablock equations can also be obtained as follows. Rewrite equation (*) as

$$G = n\hat{\mu} + E_{1b}K\hat{\beta} + E_{1v}R\hat{\tau} \qquad \text{(i)}$$
$$B = KE_{b1}\hat{\mu} + K\hat{\beta} + N\hat{\tau} \qquad \text{(ii)}$$
$$V = RE_{v1}\hat{\mu} + N'\hat{\beta} + R\hat{\tau} \qquad \text{(iii)}$$

Pre-multiply equation (ii) by $N'K^{-1}$ as

$$N'K^{-1}B = N'K^{-1}KE_{b1}\hat{\mu} + N'K^{-1}K\hat{\beta} + N'K^{-1}N\hat{\tau}$$

and subtract it from equation (iii) as

$$V - N'K^{-1}B = (RE_{v1} - N'K^{-1}KE_{b1})\hat{\mu} + (N' - N'K^{-1}K)\hat{\beta} + (R - N'K^{-1}N)\hat{\tau}$$

or $\quad V - N'K^{-1}B = [R - N'K^{-1}N]\hat{\tau}$.

Next, pre-multiply equation (ii) by NR^{-1} as

$$NR^{-1}V = NR^{-1}RE_{v1}\hat{\mu} + NR^{-1}N'\hat{\beta} + NR^{-1}R\hat{\tau}$$

and subtract it from equation (ii) as

$$B - NR^{-1}V = (KE_{b1} - NR^{-1}RE_{v1})\hat{\mu} + (K - NR^{-1}N')\hat{\beta} + (N - NR^{-1}R)\hat{\tau}$$

or $\quad B - NR^{-1}V = [K - NR^{-1}N']\hat{\beta}$

The reduced normal equation in the treatment effects can be written as

$Q = C\hat{\tau}$

where

$Q = V - N'K^{-1}B$

$C = R - N'K^{-1}N.$

The vector Q is called as the vector of **adjusted treatment totals** since it contains the treatment totals adjusted for the block effects, the matrix C is called as ***C*-matrix**.

The C matrix is symmetric and its row sums and columns sums are zero.

To show that row sum is zero in C matrix, we proceed as follows:

Row sum:

$$\begin{aligned} CE_{v1} &= RE_{v1} - N'K^{-1}NE_{v1} \\ &= (r_1, r_2, ..., r_v)' - N'K^{-1}k^{-1} = (r_1, r_2, ..., r_v)' - N'E_{b1} \\ &= r - r = 0 \end{aligned}$$

Similarly, the column sum can also be shown to be zero.

In order to obtain the reduced normal equation for treatment effects, we first estimated the block effects from one of the normal equation and substituted it into another normal equation related to the treatment effects. This way the adjusted treatment total vector Q (which is adjusted for block effects) is obtained.

Similarly, the reduced normal equations for the block effects can be found as follows. First, estimate the treatment effects from one of the normal equations and substitute it into another normal equation related to the block effects. Then we get the **adjusted block totals** (adjusted for treatment totals).

So, similar to $Q = C\hat{\tau}$, we can obtain another equation which can be represented as

$$D\hat{\beta} = P$$

where

$$D = diag(k_1, k_2, .., k_b) - N\, diag\left(\frac{1}{r_1}, \frac{1}{r_2}, ..., \frac{1}{r_v}\right) N' = K - NR^{-1}N'$$

$$P = B - N\, diag\left(\frac{1}{r_1}, \frac{1}{r_2}, ..., \frac{1}{r_v}\right) V = B - NR^{-1}V$$

$$\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, .., \hat{\beta}_b)'$$

and P is the adjusted block totals which are obtained after removing the treatment effects

Analysis of variance table:

Under the null hypothesis $H_0 : \tau = 0$, the design is one-way analysis of variance set up with blocks as classifications. In this set up, we have the following:

$$\text{sum of squares due to blocks} = \sum_{i=1}^{b} \frac{B_i^2}{k_i} - \frac{G^2}{n}$$

$$= (B_1, B_2, ..., B_b)'\, diag\left(\frac{1}{k_1}, \frac{1}{k_2}, ..., \frac{1}{k_b}\right) \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_b \end{pmatrix} - \frac{G^2}{n}$$

$$= B'K^{-1}B - \frac{G^2}{n}$$

If y is the vector of all the observations, then

$$\text{Error sum of squares} (S_e) = \sum_i \sum_j \sum_m (y_{ijm} - \hat{\mu} - \hat{\beta}_i - \hat{\tau}_j)^2$$

$$= \sum_i \sum_j \sum_m y_{ijm} (y_{ijm} - \hat{\mu} - \hat{\beta}_i - \hat{\tau}_j) \quad \text{[Using normal equations, other terms will be zero]}$$

$$= \sum_i \sum_j \sum_m y_{ijm}^2 - \hat{\mu}G - \sum_j \hat{\tau}_j V_j - \sum_i \hat{\beta}_i B_i$$

$$= y'y - \hat{\mu}G - V'\hat{\tau} - B'\hat{\beta}.$$

Using original normal equations given by

$B = KE_{b1}\hat{\mu} + K\hat{\beta} + N\hat{\tau}$,

we have

$\hat{\beta} = K^{-1}B - E_{b1}\hat{\mu} - K^{-1}N\hat{\tau}.$

Since $G = V'E_{v1} = B'E_{b1}$, substituting $\hat{\beta}$ in S_e gives

$$
\begin{aligned}
S_e &= y'y - G\hat{\mu} - B'[K^{-1}B - E_{b1}\hat{\mu} - K^{-1}N\hat{\tau}] - V'\hat{\tau} \\
&= y'y - G\hat{\mu} - B'[K^{-1}B - E_{b1}\hat{\mu} - K^{-1}N\hat{\tau}] - V'\hat{\tau} \\
&= y'y - G\hat{\mu} - B'K^{-1}B + G\hat{\mu} + B'K^{-1}N\hat{\tau} - V'\hat{\tau} \\
&= y'y - B'K^{-1}B + (B'K^{-1}N - V')\hat{\tau} \\
&= \left(y'y - \frac{G^2}{n}\right) - \left(B'K^{-1}B - \frac{G^2}{n}\right) - (V - N'K^{-1}B)'\hat{\tau}
\end{aligned}
$$

$$
\underset{\text{Error SS}}{\underset{\downarrow}{S_e}} = \underset{\text{Total SS}}{\underset{\downarrow}{\left(y'y - \frac{G^2}{n}\right)}} - \underset{\substack{\text{Block SS} \\ \text{(unadjusted)}}}{\underset{\downarrow}{\left(B'K^{-1}B - \frac{G^2}{n}\right)}} - \underset{\substack{\text{Adjusted treatment SS} \\ \text{(adjusted for blocks)}}}{\underset{\downarrow}{Q'\hat{\tau}}}
$$

The degrees of freedom associated with the different sum of squares are as follows:

Block SS (unadjusted) : $b - 1$

Treatment SS (adjusted) : $v - 1$

Error SS : $n - b - v + 1$

Total SS : $n - 1$

The adjusted treatment sum of squares and the sum of squares due to error are independently distributed and follow a Chi-square distribution with $(v-1)$ and $(n-b-v+1)$ degrees of freedom, respectively.

The analysis of variance table for $H_0 : \tau = 0$ is as follows:

Source of variation	Degrees of freedom	Sum of squares	Mean sum of squares	F
Treat	$v-1$	$Q'\hat{\tau}$ (Adjusted)	$\frac{Q'\hat{\tau}}{v-1}$	$F = \frac{Q'\hat{\tau}/(v-1)}{S_e/(n-b-v+1)}$
Blocks	$b-1$	$B'K^{-1}B - \frac{G^2}{n}$ (Unadjusted)		
Error	$n-b-v+1$	S_e	$\frac{S_e}{n-b-v+1}$	
Total	$n-1$	$y'y - \frac{G^2}{n}$		

Under H_0, $\frac{Q'\hat{\tau}/(v-1)}{S_e/(n-b-v+1)} \sim F(v-1, n-b-v+1)$.

Thus in an incomplete block design, it matters whether we are estimating the block effects first and then the treatment effects are estimated

or

first estimate the treatment effects and then the block effects are estimated.

In complete block designs, it doesn't matter at all. So the testing of hypothesis related to the block and treatment effects can be done from the same estimates.

A reason for this is as follows: In an incomplete block design, either the

- Adjusted sum of squares due to treatments, the unadjusted sum of squares due to blocks and the corresponding sum of squares due to errors are orthogonal

  or

- Adjusted sum of squares due to blocks, the unadjusted sum of squares due to treatments and the corresponding sum of squares due to errors are orthogonal.

Note that the adjusted sum of squares due to treatment and the adjusted sum of squares due to blocks are not orthogonal. So

either

Error S.S =Total SS– SS block (Unadjusted) – SS treat (Adjusted)

holds true

or

Error S.S = Total SS – SS block (Adjusted) – SS treat (Unadjusted)

holds true due to Fisher Cochran theorem.

Since $CE_{v1} = 0,$ so C is a rank deficient matrix. Also, since

$$\begin{aligned}
Q'E_{v1} &= V'E_{v1} - (N'K^{-1}B)'E_{v1} \\
&= (V_1, \ldots, V_v)E_{v1} - B'K^{-1}NE_{v1} \\
&= (\sum_i V_i) - B'K^{-1}k' \\
&= \sum_i V_i - (B_1, \ldots, B_b)\, diag\left(\frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_b}\right)\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_b \end{pmatrix} \\
&= \sum_i V_i - (B_1 \ldots\ldots B_b)E_{b1} \\
&= \sum_i V_i - \sum_j B_j \\
&= G - G \\
&= 0
\end{aligned}$$

so the intrablock equations are consistent.

We will confine our attention to those designs for which *rank*(C) = v - 1. These are called **connected designs** and for which all contrasts in the treatments, i.e., all linear combinations $\ell'\tau$ where $l'E_{v1} = 0$ have unique least-squares solutions. This we prove now as follows.

Let G^* and H^* be any two generalized inverses of C by which we mean that they are the square matrix of order v such that G^*Q and H^*Q are both the solution vectors to the intrablock equation, i.e., $\hat{\tau} = G^*Q$ and $\hat{\tau} = H^*Q$, respectively.

Then

$Q = C\hat{\tau}$

$\Rightarrow Q = CG^*Q$

and $Q = CH^*Q$ for all Q

so that $C(G^* - H^*)Q = 0.$

It follows that $(G^* - H^*)Q$ can be written as $a^* E_{v1}$ where a is any scalar which may be zero.

Let ℓ be a vector such that $\ell' E_{v1} = 0.$ The two estimates of $\ell'\tau$ are $l'G^*Q$ and $l'H^*Q$ but

$$\begin{aligned}\ell'G^*Q - \ell'H^*Q &= \ell'(G^* - H^*)Q \\ &= \ell' a^* E_{v1} \\ &= a^* \ell' E_{v1} \\ &= 0\end{aligned}$$

$\Rightarrow$ $l'\tau$ is unique.

Theorem: The adjusted treatment totals are orthogonal to the block totals.

Proof: It is enough to prove that

$$Cov(B_i, Q_j) = 0 \text{ for all } i, j.$$

Now

$$\begin{aligned}Cov(B_i, Q_j) &= Cov\left[B_i, V_j - \sum_i \left(\frac{n_{ij}}{k_i}\right) B_i\right] \\ &= Cov(B_i, V_j) - \frac{n_{ij}}{k_i} Var(B_i)\end{aligned}$$

because the block totals are mutually orthogonal, see how:

For $y_{11}, y_{12}, ..., y_{1v}$, the block total $B_1 = \sum_{j=1}^{v} y_{1j}$.

For $y_{21}, y_{22}, ..., y_{2v}$, the block total $B_2 = \sum_{j=1}^{v} y_{2j}$.

$Var(B_1) = \sum_{j=1}^{v} Var(y_{1j}) = v\sigma^2$ as $Cov(y_{1j}, y_{1k}) = 0$ for $j \neq k$

$Var(B_2) = \sum_{j=1}^{v} Var(y_{2j}) = v\sigma^2$ as $Cov(y_{2j}, y_{2k}) = 0$ for $j \neq k$

$Var(B_1) + Var(B_2) = 2v\sigma^2$ as $Cov(y_{1j}, y_{2k}) = 0$ for $j \neq k$

$\Rightarrow$ B_1 and B_2 are mutually orthogonal as all y_{ij}'s are independent.

As B_i and V_j have n_{ij} observations in common and the observations are mutually independent, so

$$Cov(B_i, V_j) = n_{ij}\sigma^2$$
$$Var(B_i) = k_i\sigma^2$$

Thus $Cov(B_i, Q_j) = n_{ij}\sigma^2 - \dfrac{n_{ij}}{k_i}k_i\sigma^2 = 0$

Hence proved.

Theorem:

$$E(Q) = C\tau$$
$$Var(Q) = \sigma^2 C$$

Proof:

$$Q_j = V_j - \left[\frac{n_{1j}B_1}{k_1} + ... + \frac{n_{bj}B_b}{k_b}\right]$$
$$= V_j - \sum_{i=1}^{b}\frac{n_{ij}B_i}{k_i}$$
$$E(Q_j) = E(V_j) - \sum_{i=1}^{b}\frac{n_{ij}}{k_i}E(B_i)$$
$$E(V_j) = \sum_i\sum_m E(y_{ijm})$$
$$= \sum_i\sum_m E(\mu + \beta_i + \tau_j + \varepsilon_{ijm})$$
$$= \mu\sum_i n_{ij} + \sum_i \beta_i n_{ij} + \tau_j\sum_i n_{ij}$$
$$= \mu r_j + \sum_i \beta_i n_{ij} + \tau_j r_j$$

$$E(B_i) = \sum_j\sum_m E(y_{ijm})$$
$$= \sum_j\sum_m E(\mu + \beta_i + \tau_j + \varepsilon_{ijm})$$
$$= \sum_j\sum_m (\mu + \beta_i + \tau_j)$$
$$= \mu k_i + \beta_i k_i + \sum_j \tau_j n_{ij}$$

$$\sum_{i=1}^{b}\frac{n_{ij}}{k_i}E(B_i) = \sum_{i=1}^{b}\frac{n_{ij}}{k_i}\left[\mu k_i + \beta_i k_i + \sum_j \tau_j n_{ij}\right]$$
$$= \mu r_j + \sum_i \beta_i n_{ij} + \sum_i \frac{n_{ij}}{k_i}\left(\sum_j \tau_j n_{ij}\right).$$

Thus substituting these expressions in $E(Q_j)$, we have

$$E(Q_j) = r_j\tau_j - \sum_i \frac{n_{ij}}{k_i}(\sum_j \tau_j n_{ij})$$

$$= \left(r_i - \sum_i \frac{n_{ij}^2}{k_i} \right)\tau_j - \sum_i \frac{n_{ij}}{k_i} \sum_{j(\neq \ell)} n_{i\ell}\tau_\ell = c_{jj}\tau_j + \sum_{j'(\neq \ell)} c_{\ell j'}\tau_{j'}$$

Further, substituting $E(Q_j)$ in $E(Q) = (E(Q_1), E(Q_2), ..., E(Q_b))'$, we get

$$E(Q) = C\tau$$

Next

$$Var(Q) = \begin{pmatrix} Var(Q_1) & Cov(Q_1,Q_2) & ... & Cov(Q_1,Q_v) \\ Cov(Q_2,Q_1) & Var(Q_2) & ... & Cov(Q_2,Q_v) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(Q_v,Q_1) & Cov(Q_v,Q_2) & ... & Var(Q_v) \end{pmatrix}$$

$$Var(Q_j) = Var[V_j - \sum_i \frac{n_{ij}}{k_i} B_i]$$

$$= Var(V_j) + \sum_i \left(\frac{n_{ij}}{k_i} \right)^2 Var(B_i) - 2\sum_i \frac{n_{ij}}{k_i} Cov(V_j, B_i),$$

Note that

$$Var(V_j) = Var(\sum_i \sum_m y_{ijm}) = r_j\sigma^2$$

$$Var(B_i) = Var(\sum_j \sum_m y_{ijm}) = k_i\sigma^2$$

$$Cov(V_j, B_i) = Cov\left(\sum_i \sum_m y_{ijm}, \sum_j \sum_m y_{ijm} \right) = n_{ij}\sigma^2$$

$$Var(Q_j) = r_j\sigma^2 + \sum_i \left(\frac{n_{ij}}{k_i} \right)^2 k_i\sigma^2 - 2\sum_i \left(\frac{n_{ij}}{k_i} \right) n_{ij}\sigma^2$$

$$= r_j\sigma^2 - \sum_i \frac{n_{ij}^2}{k_i}\sigma^2 - 2\sum_i \left(\frac{n_{ij}^2}{k_i} \right)\sigma^2$$

$$= r_j\sigma^2 - \sum_i \left(\frac{n_{ij}^2}{k_i} \right)\sigma^2$$

$$= c_{jj}\sigma^2.$$

$$Cov(Q_j, Q_\ell) = Cov\left[V_j - \sum_i \frac{n_{ij}}{k_i} B_i,\ V_\ell - \sum_i \frac{n_{i\ell}}{k_i} B_i\right]$$

$$= Cov(V_j, V_\ell) - \sum_i \frac{n_{i\ell}}{k_i} Cov(V_j, B_i) - \sum_i \frac{n_{ij}}{k_i} Cov(B_i, V_\ell) + \sum_i \frac{n_{ij} n_{i\ell}}{k_i^2} Cov(B_i, B_i)$$

$$= 0 - \sum_i \frac{n_{i\ell}}{k_i} Cov(V_j, B_i) - \sum_i \frac{n_{ij}}{k_i} Cov(B_i, V_\ell) + \sum_i \frac{n_{ij} n_{i\ell}}{k_i^2} Var(B_i)$$

$$= -\sum_i \left(\frac{n_{i\ell} n_{ij}}{k_i}\right)\sigma^2 - \sum_i \left(\frac{n_{ij} n_{i\ell}}{k_i}\right)\sigma^2 + \sum_i \frac{n_{ij} n_{i\ell}}{k_i^2} k_i \sigma^2$$

$$= c_{j\ell}\sigma^2$$

Substituting the terms of $Var(Q_j) = c_{jj}\sigma^2$ and $Cov(Q_j, Q_l) = c_{jl}\sigma^2$ in $Var(Q),$ we get $Var(Q) = C\sigma^2.$

Hence proved.

[Note: We will prove this result using the matrix approach later].

Covariance matrix of adjusted treatment totals:

Consider $Z = \begin{pmatrix} V \\ B \end{pmatrix}$ with *b* + *v* variables.

We can express

$$Q = V - N'K^{-1}B$$

$$= [I \quad -N'K^{-1}]\begin{bmatrix} V \\ B \end{bmatrix}$$

$$= [I \quad -N'K^{-1}]Z.$$

So

$$Cov(Q) = [I \quad -N'K^{-1}]\ Cov(Z)\begin{bmatrix} I' \\ (-N'K^{-1})' \end{bmatrix}.$$

Now we find that

$$Cov(Z) = \begin{pmatrix} Var(V) & Cov(V,B) \\ Cov(B,V) & Var(B) \end{pmatrix}$$

Since B_i and V_j have n_{ij} observations in common and the observations are mutually independent, so

$$Cov(B_i, V_j) = n_{ij}\sigma^2$$

$$Var(B_i) = k_i\sigma^2$$

$$Var(V_j) = r_j\sigma^2.$$

Thus

$$Cov(Z) = \begin{pmatrix} R & N' \\ N & K \end{pmatrix} \sigma^2$$

$$\begin{aligned} Cov(Q) &= [I \quad -N'K^{-1}] \begin{bmatrix} R & N' \\ N & K \end{bmatrix} \begin{bmatrix} I \\ -K^{-1}N \end{bmatrix} \sigma^2 \\ &= [R - N'K^{-1}N \quad N' - N'] \begin{bmatrix} I \\ -K^{-1}N \end{bmatrix} \sigma^2 \\ &= (R - N'K^{-1}N)\sigma^2 \\ &= C\sigma^2. \end{aligned}$$

Next, we show that $Cov(B,Q) = 0$

$$\begin{aligned} Cov(B,Q) &= Cov(B,V) - Cov(B, V - N'K^{-1}B) \\ &= Cov(B,V) - Var(B)K^{-1}N \\ &= N\sigma^2 - KK^{-1}N\sigma^2 \\ &= 0. \end{aligned}$$

An alternative approach to find/ prove $E(Q) = C\tau,\ D(Q) = C\sigma^2$

Now we illustrate another approach to find the expectations etc in the set up of an incomplete block design. We have now learnt three approaches- the classical approach based on summations, the approach based on matrix theory and this new approach which is also based on the matrix theory. We can choose any of the approaches. The objective here is to let the reader know these different approaches.

Rewrite the linear model

$$y_{ijm} = \mu + \beta_i + \tau_j + \varepsilon_{ijm}, \quad i = 1,2,...,b;\ j = 1,2,...,v; m = 0,1,..,n_{ij}.$$

as

$$y = \mu E_{n1} + D_1'\tau + D_2'\beta + \varepsilon$$

where

$$\tau = (\tau_1, \tau_2, ..., \tau_v)'$$
$$\beta = (\beta_1, \beta_2, ..., \beta_b)'.$$

Since B_i and V_j have n_{ij} observations in common and the observations are mutually independent, so denote

$D_1 : v \times n$ matrix of treatment effect versus N, i.e., $(i,\ j)^{\text{th}}$ element of this matrix is given by

$$D_1 = \begin{cases} 1 & \text{if } j^{th} \text{ observation comes from } i^{th} \text{ treatment} \\ 0 & \text{otherwise} \end{cases}$$

$D_2 : b \times n$ matrix of block effects versus N, i.e., $(i,\ j)^{\text{th}}$ element of this matrix is given by

$$D_2 = \begin{cases} 1 & \text{if } j^{th} \text{ observation comes from } i^{th} \text{ block} \\ 0 & \text{otherwise.} \end{cases}$$

Following results can be verified:

$$D_1 D_1' = R = diag(r_1, r_2, ..., r_v),$$
$$D_2 D_2' = K = diag(k_1, k_2, ..., k_b),$$
$$D_2 D_1' = N \text{ or } D_1 D_2' = N'$$
$$D_1 E_{n1} = (r_1, r_2, ..., r_v)'$$
$$D_2 E_{n1} = (k_1, k_2, ..., k_b)'$$
$$D_1' E_{v1} = E_{n1} = D_2' E_{b1}.$$

In earlier notations,

$$V = (V_1, V_2, ..., V_v)' = D_1 y$$
$$B = (B_1, B_2, ..., B_b)' = D_2 y$$

Express Q in terms of D_1 and D_2 as

$$\begin{aligned} Q &= V - N'K^{-1}B \\ &= \left[D_1 - D_1 D_2' (D_2 D_2')^{-1} D_2 \right] y. \end{aligned}$$

Then

$$\begin{aligned} E(Q) &= \left[D_1 - D_1 D_2' (D_2 D_2')^{-1} D_2 \right] E(y) \\ &= \left[D_1 - D_1 D_2' (D_2 D_2')^{-1} D_2 \right] (\mu E_{n1} + D_1' \tau + D_2' \beta) \\ &= \left[D_1 E_{n1} - D_1 D_2' (D_2 D_2')^{-1} D_2 E_{n1} \right] \mu + \left[D_1 D_1' - D_1 D_2' (D_2 D_2')^{-1} D_2 D_1' \right] \tau \\ &\quad + \left[D_1 D_2' - D_1 D_2' (D_2 D_2')^{-1} D_2 D_2' \right] \beta \\ &= \left[(r_1, r_2, ..., r_v)' - N'K^{-1}(k_1, ..., k_b)' \right] \mu + \left[R - N'K^{-1}N \right] \tau + \left[N' - N'K^{-1}K \right] \beta. \end{aligned}$$

Since $N'K^{-1}(k_1,...,k_b)' = N' diag\left(\frac{1}{k_1},\frac{1}{k_2},...,\frac{1}{k_b}\right)\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_b \end{pmatrix}$

$$= N'\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n_{11} & n_{21}... \, n_{b1} \\ n_{12} & n_{22}... \, n_{b2} \\ \vdots & \vdots \; \ddots \, \vdots \\ n_{iv} & n_{2v}...n_{bv} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \left(\sum_{i=1}^{b} n_{i1},\ \sum_{i=1}^{b} n_{i2},...,\sum_{i=1}^{b} n_{iv} \right)' = (r_1, r_2,..., r_v)'.$$

Thus

$$N' - N'K^{-1}K = \left(r_1, r_2,..., r_v\right)' - N'K^{-1}(k_1,...,k_b) = 0$$

and so

$$E(Q) = \left[R - N'K^{-1}N\right]\tau$$
$$= C\tau.$$

Next

$$\begin{aligned} Var(Q) &= D_1\left[I - D_2'(D_2D_2')^{-1}D_2\right]Var(y)\left[I - D_2'(D_2D_2')^{-1}D_2\right]D_1' \\ &= \sigma^2 D_1\left[I - D_2'(D_2D_2')^{-1}D_2\right]D_1' \\ &= \sigma^2\left[D_1D_1' - D_1D_2'(D_2D_2')^{-1}D_2D_1'\right] \\ &= \sigma^2\left[R - N'K^{-1}N\right] \\ &= \sigma^2 C. \end{aligned}$$

Note that $\left[I - D_2'(D_2D_2')^{-1}D_2\right]$ is an idempotent matrix.

Similarly, we can also express

$$P = B - NR^{-1}V$$
$$= [D_2 - D_2D_1'R^{-1}D_1]y.$$

Theorem: $E(P) = D\beta, \quad Var(P) = \sigma^2 D$

Proof:

$$D = K - NR^{-1}N'$$
$$P = B - NR^{-1}V$$
$$= D_2[I - D_1R^{-1}D_1]y$$
$$= D_2[I - D_1'(D_1D_1')^{-1}D_1]y$$

$$
\begin{aligned}
E(P) &= D_2[I_1 - D_1'(D_1D_1')^{-1}D_1](\mu E_{n1} + D_1'\tau + D_2'\beta)\\
&= [D_2E_{n1} - D_2D_1'R^{-1}D_1E_{n1}]\mu + [D_2D_1' - D_2D_1'R^{-1}D_1D_1']\tau\\
&\quad + [D_2D_1' - D_2D_1'R^{-1}D_1D_2']\beta\\
&= [(k_1, k_2, ..., k_b)' - NR^{-1}(r_1, r_2, ..., r_v)']\mu + [N - NR^{-1}R]\tau + [K - NR^{-1}N']\beta\\
&= [(k_1, k_2, ..., k_b)' - NE_{v1}]\mu + 0 + D\beta\\
&= [(k_1, k_2, ..., k_b)' - (k_1, k_2, ..., k_b)']\mu + D\beta\\
&= D\beta
\end{aligned}
$$

Next

$$
\begin{aligned}
Var(P) &= \sigma^2 D_2[I - D_1'(D_1D_1')^{-1}D_1]D_2'\\
&= \sigma^2[D_2D_2' - D_2D_1'(D_1D_1')^{-1}D_1D_2']\\
&= \sigma^2[K - NR^{-1}N'] = \sigma^2 D
\end{aligned}
$$

Note that $[I - D_1'(D_1D_1')^{-1}D_1]$ is an idempotent matrix.

Alternatively, we can also find $Var(P)$ as follows:

$$
P = (I \quad -NR^{-1})\begin{pmatrix} B \\ V \end{pmatrix} = (I \quad -NR^{-1})Z \quad \text{where } Z = (B, \quad V)'
$$

$$
\begin{aligned}
Var(P) &= (I \quad -NR^{-1})Cov(Z)\begin{pmatrix} I \\ -R^{-1}N' \end{pmatrix}\\
&= (I \quad -NR^{-1})\begin{pmatrix} K & N \\ N' & R \end{pmatrix}\begin{pmatrix} I \\ -R^{-1}N' \end{pmatrix}\sigma^2\\
&= (K - NR^{-1}N' \quad N - NR^{-1}R)\begin{pmatrix} I \\ -R^{-1}N' \end{pmatrix}\sigma^2\\
&= \left(K - NR^{-1}N'\right)\sigma^2\\
&= D\sigma^2
\end{aligned}
$$

Now we consider some properties of incomplete block designs.

Lemma: $b + rank(C) = v + rank(D)$.

Proof: Consider $(b + v)$x$(b + v)$ matrix

$$
A = \begin{bmatrix} K & N \\ N' & R \end{bmatrix}
$$

Note that A is a submatrix of C.

Using the result that the rank of a matrix does not change by the pre-multiplication of the nonsingular matrix, consider the following matrices:

$$M = \begin{bmatrix} I_b & 0 \\ -N'K^{-1} & I_v \end{bmatrix}$$

and

$$S = \begin{bmatrix} I_b & 0 \\ -R^{-1}N' & I_v \end{bmatrix}.$$

M and *S* are nonsingular, so we have

rank(*A*) = *rank*(*MA*) = *rank*(*AS*).

Now

$$MA = \begin{bmatrix} I_b & 0 \\ -N'K^{-1} & I_v \end{bmatrix} \begin{bmatrix} K & N \\ N' & R \end{bmatrix} = \begin{bmatrix} K & N \\ 0 & C \end{bmatrix}$$

$$AS = \begin{bmatrix} D & N \\ 0 & R \end{bmatrix}.$$

Thus

$$rank \begin{bmatrix} K & N \\ 0 & C \end{bmatrix} = rank \begin{bmatrix} D & N \\ 0 & R \end{bmatrix}$$

or

$$rank(K) + rank(C) = rank(D) + rank(R)$$

or

$$b + rank(C) = v + rank(D)$$

Remark:

$C : v \times v$ and $D : b \times b$ are symmetric matrices.

One can verify that

$$CE_{v1} = 0$$

and $\quad DE_{b1} = 0$

Thus $\quad rank(C) \leq v - 1$

and $\quad rank(D) \leq b - 1.$

Lemma:

If *rank*(*C*) = *v* - 1, then all blocks and treatment contrasts are estimable.

Proof:

If *rank*(*C*) = *v* – 1, it is obvious that all the treatment contrasts are estimable.

Using the result from the lemma $b + rank(C) = v + rank(D)$, we have

$rank(D) + v = rank(C) + b$

$= v - 1 + b$

Thus

$rank(D) = b - 1.$

Thus all the block contrasts are also estimable.

Orthogonality of *Q* and *P*:

Now we explore the conditions under which Q and P can be orthogonal.

$$\begin{aligned}
Q &= V - N'K^{-1}B \\
&= (D_1 - D_1 D_2' K^{-1} D_2) y \\
P &= B - NR^{-1}V \\
&= (D_2 - D_2 D_1' R^{-1} D_1) y \\
Cov(Q,P) &= (D_1 - D_1 D_2' K^{-1} D_2)(D_2 - D_2 D_1' R^{-1} D_1)' \sigma^2 \\
&= (D_1 D_2' - D_1 D_2' R^{-1} D_1 D_2' - D_1 D_2' K^{-1} D_2 D_2' + D_1 D_2' K^{-1} D_2 D_1' R^{-1} D_1 D_2') \sigma^2 \\
&= (N' - RR^{-1}N' - N'K^{-1}K + N'K^{-1}NR^{-1}N') \sigma^2 \\
&= (N'K^{-1}NR^{-1}N' - N') \sigma^2
\end{aligned}$$

Q and P (or equivalently Q_i and P_j) are orthogonal when

$$Cov(Q,P) = 0$$

or $N'K^{-1}N\ R^{-1}N' - N' = 0$ (i)

$\Rightarrow (R - C)R^{-1}N' - N' = 0$ (Using $C = R - N'K^{-1}N$)

$\Rightarrow CR^{-1}N' = 0$ (ii)

or equivalently

$$N'K^{-1}NR^{-1}N' - N' = 0$$

$\Rightarrow N'K^{-1}(K - D) - N' = 0$ (Using $D = K - NR^{-1}N'$)

$\Rightarrow N'K^{-1}D = 0$ (iii)

Thus Q_i and P_j are orthogonal if

$NR^{-1}N'K^{-1}N = N'$

or equivalently

$NR^{-1}C = 0$

or equivalently

$DK^{-1}N = 0.$

Orthogonal block design:

A block design is said to be orthogonal if Q_i's and P_j's are orthogonal for all *i* and *j*. Thus the condition for the orthogonality of design is $NR^{-1}N'K^{-1}N = N$, $NR^{-1}C = 0$ or $DK^{-1}N = 0$.

Lemma: If $\frac{n_{ij}}{r_j}$ is constant for all *j*, then $\frac{n_{ij}}{k_i}$ is constant for all *i* and vice versa. In this case, we have

$$n_{ij} = \frac{k_i r_j}{n}.$$

Proof. If $\frac{n_{ij}}{r_j}$ is constant for all *j* then $\frac{n_{ij}}{r_j} = a_i$, say.

$$\Rightarrow n_{ij} = a_i r_j$$

$$\text{or } \sum_j n_{ij} = \sum_j a_i r_j = a_i \sum_j r_j = a_i n$$

$$\text{or } k_i = a_i n$$

$$\text{or } a_i = \frac{k_i}{n}$$

Thus

$$\frac{n_{ij}}{r_j} = \frac{k_i}{n}$$

$$\text{or } n_{ij} = \frac{k_i r_j}{n}$$

$$\text{So } \frac{n_{ij}}{k_j} = \frac{r_j}{n} : \text{independent of } i.$$

Hence proved.

Contrast:

A linear function $\sum_{j=1}^{v} c_j \tau_j = C'\tau$ where $c_1, c_2, ..., c_v$ are given number such that $\sum_{j=1}^{v} c_j = 0$ is called a contrast of τ_j's.

Elementary contrast:

A contrast $\sum_{j=1}^{v} c_j \tau_j = C'\tau$ with $C = (c_1, c_2, ..., c_v)'$ in treatment effects $\tau = (\tau_1, \tau_2, ..., \tau_v)'$ is called an elementary contrast if *C* has only two non-zero components 1 and -1.

Elementary contrasts in the treatment effects involve all the differences in the form $\tau_i - \tau_j$, $i \neq j$.

It is desirable to design experiments where all the elementary contrasts are estimable.

Connected Design:

A design where all the elementary contrasts are estimable is called a connected design otherwise it is called a **disconnected design**.

The physical meaning of connectedness of a design is as follows:

Given any two treatment effects τ_{i1} and τ_{i2}, it is possible to have a chain of treatment effects like $\tau_{i1}, \tau_{1j}, \tau_{2j}, ..., \tau_{nj}, \tau_{i2}$, such that two adjoining treatments in this chain occur in the same block.

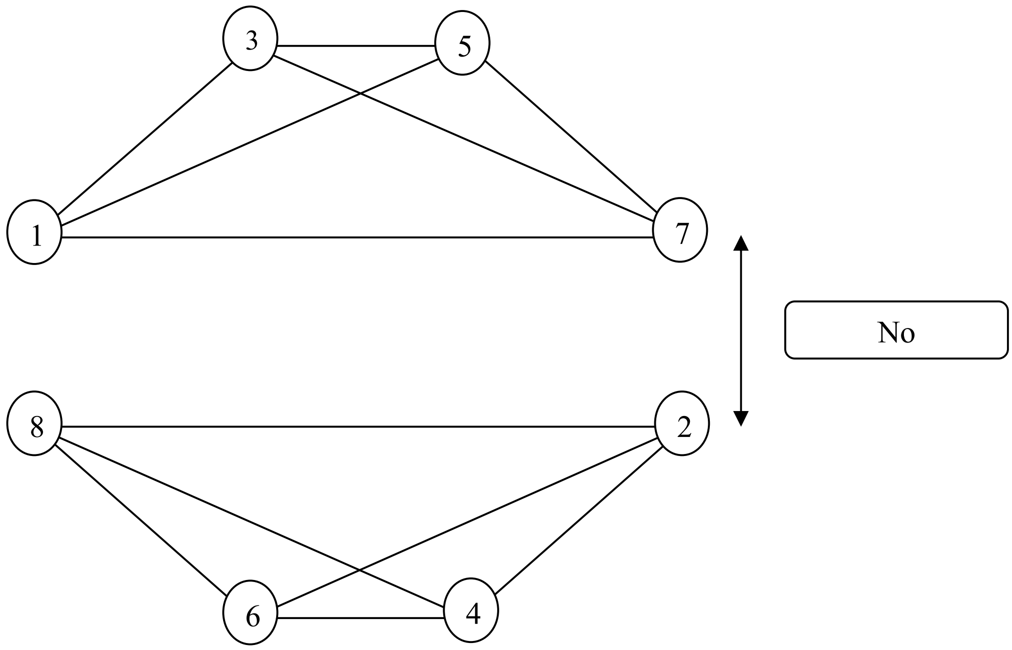
Example of connected design:

In a connected design, within every block, all the treatment contrasts are estimable and pair-wise comparison of estimators have similar variances.

Consider a disconnected incomplete block design as follows:
b = 8 (Block numbers: I, II,…,VIII), k = 3, v = 8 (treatment numbers: 1,2,…,8), r = 3

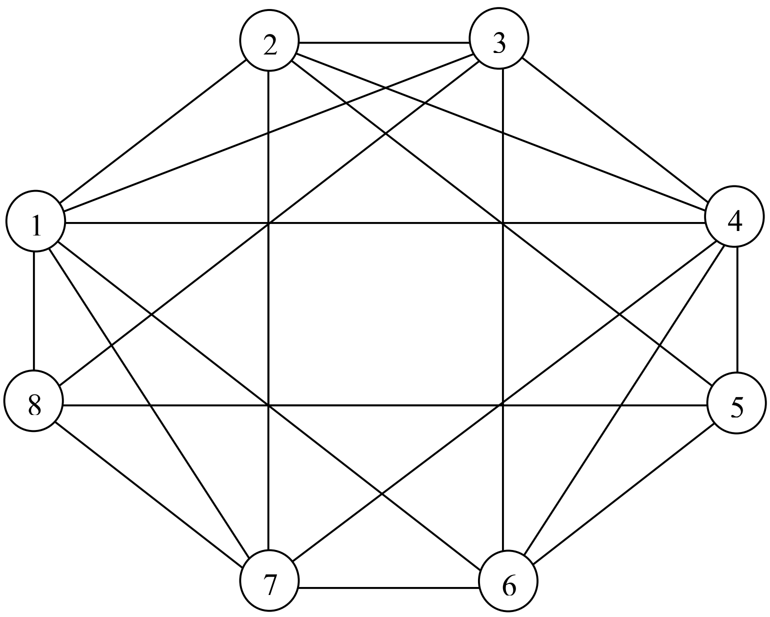
Blocks	Treatments
I	1 3 5
II	2 4 6
III	3 5 7
IV	4 6 8
V	5 7 1
VII	6 8 2
VII	7 1 3
VIII	8 2 4

The blocks of this design can be represented graphically as follows:



Note that it is not possible to reach the treatment, e.g., 7 from 2, 3 from 4 etc. So the design is not connected.

Moreover, if the blocks of the design are given like in the following figure, then any treatment can be reached from any treatment. So the design, in this case, is connected. For example, treatment 2 can be reached from treatment 6 through different routes like $6 \to 5 \to 4 \to 3 \to 2,\ 6 \to 3 \to 2,\ 6 \to 7 \to 8 \to 1 \to 2, 6 \to 7 \to 2$ etc.



A design is connected if every treatment can be reached from every treatment via lines in the connectivity graph.

Theorem: An incomplete block design with v treatments is connected if and only if $rank(C) = v - 1$.

Proof: Let the design be connected. Consider a set of (v - 1) linearly independent contrasts $\tau_i - \tau_j$ (j = 2, 3,…,v). Let these contrasts be $C_2'\tau, C_3'\tau, ..., C_v'\tau$ where $\tau = (\tau_{.1}, \tau_{.2}, ...\tau_v)'$. Obviously, vectors $C_2, C_3, ..., C_v$ form the basis of vector space of dimension (v - 1). Thus any contrast $p'\tau$ is expressible as a linear combination of the contrasts $C_i'\tau (i = 2, 3, ..., v)$. Also $p'\tau$ is estimable if and only if p belongs to the column space of C-matrix of the design.

Therefore, the dimension of column space of C must be the same as that of the vector space spanned by the vectors $C_i (i = 2, 3, ..., v)$, i.e., equal to (v - 1).

Thus $rank(C) = v - 1$.

Conversely, let $rank(C) = v - 1$ and let $\xi_1, \xi_2, ..., \xi_{v-1}$ be a set of orthonormal eigenvectors corresponding to the (not necessarily distinct) non-zero eigenvalues $\theta_1, \theta_2, ..., \theta_{v-1}$ of C.

Then

$$E(\xi_1' Q) = \xi_i' C\tau$$
$$= \theta_i \xi_i' \tau.$$

Thus an unbiased estimator of $\xi_i'\tau$ is $\dfrac{\xi_i' Q}{\theta_i}$.

Also, since each ξ_i is orthogonal to E_{v1} and $\xi_i' s$ are mutually orthogonal, so any contrast $p'\tau$ belongs to the vector space spanned by $\{\xi_i, i = 1...v\}$, *i.e.*, $p = \sum_{i=1}^{v-1} a_i \xi_i$.

So $E\left[\sum_{i=1}^{v-1} a_i \dfrac{\xi_i Q}{\theta_i}\right] = p'\tau$.

Thus $p'\tau$ is estimable and this completes the proof.

Lemma: For a connected block design $Cov(Q,P)=0$ if and only if $N'=\frac{rk'}{n}.$

Proof: "if" part

When $N'=\frac{rk'}{n},$ we have

$$\frac{1}{\sigma^2}Cov(Q,P)=N'K^{-1}NR^{-1}N'-N'$$
$$=\frac{rk'K^{-1}kr'R^{-1}N'}{n^2}-N'$$

Since $k'K^{-1}k=(k_1,k_2,...,k_b)diag\left(\frac{1}{k_1},\frac{1}{k_2},...,\frac{1}{k_b}\right)\begin{pmatrix}k_1\\k_2\\\vdots\\k_b\end{pmatrix}$

$$=(1,...,1)\begin{pmatrix}k_1\\\vdots\\k_b\end{pmatrix}=\sum_{i=1}^{b}k_i=n$$

and

$$r'R^{-1}=(r_1,r_2,...,r_v)diag\left(\frac{1}{r_1},\frac{1}{r_2},...,\frac{1}{r_v}\right)=E_{1v}.$$

Then

$$\frac{1}{\sigma^2}Cov(Q,P)=\frac{rnE_{1v}N'}{n^2}-N'$$
$$=\frac{rE_{1v}N'}{n}-N'=\frac{rk'}{n}-N'$$
$$=N'-N'=0.$$

"Only if part"

Let $Cov(Q,P)=0$

$\Rightarrow N'K^{-1}NR^{-1}N'-N'=0$ (Since $C=R-N'K^{-1}N$)

or $(R-C)R^{-1}N'-N'=0$

or $CR^{-1}N'=0.$

Let

$R^{-1}N'=A=(\underline{a}_1,\underline{a}_2,...,\underline{a}_b)$

where $\underline{a}_1,\underline{a}_2,...,\underline{a}_b$ are the columns of A.

Since the design is connected, so the columns of A are proportional to E_{v1}. Also, all row/column sums of C are zero.

So $(CE_{v1}, CE_{v1}, ..., CE_{v1}) = 0$

and

$CA = 0$

or $C(\underline{a}_1, \underline{a}_2, ..., \underline{a}_b) = 0$

$\Rightarrow a_i \propto E_{v1}$

or $a_i = \alpha_i E_{v1}$; $i = 1, 2, ..., b$

where α_i are some scalars.

This gives

$A = R^{-1}N' = E_{v1}\alpha'$ where $\alpha = (\alpha_1 ....\alpha_b)'$.

So we have

$$N' = RE_{v1}\alpha = (r_1, r_2, .., r_v)'\alpha'$$
$$= r\alpha' \quad \text{where } r = (r_1, r_2, .., r_v)'.$$

Pre-multiply by E_{1v} gives

$E_{1v}N' = (k_1, k_2, ..., k_b)' = E_{1v}r\alpha' = n\alpha'$

or $k = n\alpha'$

$\Rightarrow \alpha' = \dfrac{k'}{n}$ where $k = (k_1, k_2, ..., k_b)'$

Thus

$$N' = r\alpha' = \frac{rk'}{n}.$$

Hence proved.

Definition: A connected block design is said to be **orthogonal** if and only if the incidence matrix of the design satisfies the condition $N' = \dfrac{rk'}{n}$.

Designs which do not satisfy this condition are called **non-orthogonal**. It is clear from this result that if at least one entry of N is zero, the design cannot be orthogonal.

A block design with at least one zero-entry in its incidence matrix is called an incomplete block design.

Theorem: A sufficient condition for the orthogonality of design is that $\frac{n_{ij}}{r_j}$ is constant for all j.

Conclusion: It is obvious from the condition of orthogonality of a design that a design which is not connected and an incomplete design even though it may be connected cannot have an orthogonal structure.

Now we illustrate the general nature of the incomplete block design. We try to obtain the results for a randomized block design through the results of an incomplete block design.

Randomized block design:

The randomized block design is an arrangement of v treatment in b blocks of v plots each, such that every treatment occurs in every block, one treatment in each plot.

The arrangement of treatment within a block is random and in terms of incidence matrix,

$n_{ij} = 1$ for all $i = 1, 2, ..., b;\ j = 1, 2, ..., v.$

Thus we have

$$k_i = \sum_j n_{ij} = v \text{ for all } i$$

$$r_j = \sum_j n_{ij} = b \text{ for all } j.$$

We have $\frac{n_{ij}}{r_j} = \frac{1}{b}$ constant for all j.

$$C_{jj} = b - \frac{b}{v}$$

$$C_{jj'} = -\frac{b}{v}$$

$$Q_j = V_j - \frac{G}{v}.$$

Normal equations for $\tau's$ are

$$\left(b - \frac{b}{v}\right)\tau_j - \frac{b}{v}\sum_{i \neq j'=1}^{v} \tau_{j'} = V_j - \frac{G}{v};\ j = 1....v$$

$$\tau_1 + \tau_2 .... + \tau_v = 0.$$

Thus

$$b\tau_j - \frac{b}{v}\sum_j \tau_j = V_j - \frac{G}{v}$$

$$\text{or } \hat{\tau}_j = \frac{1}{b}\left(V_j - \frac{G}{v}\right) = \bar{y}_{oj} - \bar{y}_{oo}.$$

The sum of squares due to treatments adjusted for blocks is

$$= \sum_j \hat{\tau}_j Q_j$$

$$= \frac{1}{b}\sum_j \left(V_j - \frac{G}{v}\right)^2$$

$$= \frac{\sum_j V_j^2}{b} - \frac{G^2}{bv},$$

which is also the sum of squares due to treatments which are unadjusted for blocks because the design is orthogonal.

$$\text{Sum of squares due to blocks} = \frac{\sum_i B_i^2}{v} - \frac{G^2}{bv}$$

$$\text{Sum of squares due to error} = \sum_i \sum_j \left(y_{ij} - \frac{B_i}{v} - \frac{V_j}{b} + \frac{G}{bv}\right)^2.$$

These expressions are the same as obtained under the analysis of variance in the set up of a randomized block design.

Interblock analysis of incomplete block design

The purpose of block designs is to reduce the variability of response by removing the part of the variability as block numbers. If in fact, this removal is illusory, the block effects being all equal, then the estimates are less accurate than those obtained by ignoring the block effects and using the estimates of treatment effects. On the other hand, if the block effect is very marked, the reduction in the basic variability may be sufficient to ensure a reduction of the actual variances for the block analysis.

In the intrablock analysis related to treatments, the treatment effects are estimated after eliminating the block effects. If the block effects are marked, then the block comparisons may also provide information about the treatment comparison. So a question arises how to utilize the block information additionally to develop an analysis of variance to test the hypothesis about the significance of treatment effects.

Such an analysis can be derived regarding the block effects as random variables. This assumption involves the random allocation of different blocks of the design to be the blocks of the material selected (at random from the population of possible blocks) in addition to the random allocation of treatments occurring in a block to the units of the block selected to contain them. Now the two responses from the same block are correlated because the error associated with each contains the block number in common. Such an analysis of incomplete block design is termed as interblock analysis.

To illustrate the idea behind the interblock analysis and how the block comparisons also contain information about the treatment comparisons, consider an allocation of four selected treatments in two blocks each. The outputs (y_{ij}) are recorded as follows:

$$\text{Block 1: } y_{14} \;\; y_{16} \;\; y_{17} \;\; y_{19}$$
$$\text{Block 2: } y_{23} \;\; y_{25} \;\; y_{26} \;\; y_{27.}$$

The block totals are

$$B_1 = y_{14} + y_{16} + y_{17} + y_{19},$$
$$B_2 = y_{23} + y_{25} + y_{26} + y_{27}.$$

Following the model $y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij},\ i = 1,2,\ j = 1,2,...,9,$ we have

$$\begin{aligned}
y_{14} &= \mu + \beta_1 + \tau_4 + \varepsilon_{14},\\
y_{16} &= \mu + \beta_1 + \tau_6 + \varepsilon_{16},\\
y_{17} &= \mu + \beta_1 + \tau_7 + \varepsilon_{17},\\
y_{19} &= \mu + \beta_1 + \tau_9 + \varepsilon_{19},\\
y_{23} &= \mu + \beta_2 + \tau_3 + \varepsilon_{23},\\
y_{25} &= \mu + \beta_2 + \tau_5 + \varepsilon_{25},\\
y_{26} &= \mu + \beta_2 + \tau_6 + \varepsilon_{26},\\
y_{27} &= \mu + \beta_2 + \tau_7 + \varepsilon_{27},
\end{aligned}$$

and thus

$$\begin{aligned}
B_1 - B_2 = {} & 4(\beta_1 - \beta_2) + (\tau_4 + \tau_6 + \tau_7 + \tau_9) - (\tau_3 + \tau_5 + \tau_6 + \tau_7)\\
& + (\varepsilon_{14} + \varepsilon_{16} + \varepsilon_{17} + \varepsilon_{19}) - (\varepsilon_{23} + \varepsilon_{25} + \varepsilon_{26} + \varepsilon_{27}).
\end{aligned}$$

If we assume additionally that the block effects β_1 and β_2 are random with mean zero, then

$$E(B_1 - B_2) = (\tau_4 + \tau_9) - (\tau_3 + \tau_5)$$

which reflects that the block comparisons can also provide information about the treatment comparisons.

The intrablock analysis of an incomplete block design is based on estimating the treatment effects (or their contrasts) by eliminating the block effects. Since different treatments occur in different blocks, so one may expect that the block totals may also provide some information on the treatments. The interblock analysis utilizes the information on block totals to estimate the treatment differences. The block effects are assumed to be random and so we consider the set up of mixed effect model in which the treatment effects are fixed but the block effects are random. This approach is applicable only when the number of blocks is more than the number of treatments. We consider here the interblock analysis of binary proper designs for which $n_{ij} = 0$ or 1 and $k_1 = k_2 = ... = k_b = k$ in connection with the intrablock analysis.

Model and Normal Equations

Let y_{ij} denotes the response from the j^{th} treatment in i^{th} block from the model

$$y_{ij} = \mu^* + \beta_i^* + \tau_j + \varepsilon_{ij}, i = 1, 2, ..., b; j = 1, 2, ..., v$$

where

μ^* is the general mean effect;

β_i^* is the random additive i^{th} block effect;

τ_j is the fixed additive j^{th} treatment effect; and

ε_{ij} is the i.i.d. random error with $\varepsilon_{ij} \sim N(0, \sigma^2)$.

Since the block effect is now considered to be random, so we additionally assume that $\beta_i^* (i = 1, 2, .., b)$ are independently distributed following $N(0, \sigma_\beta^2)$ and are uncorrelated with ε_{ij}. One may note that we can not assume here $\sum_i \beta_i^* = 0$ as in other cases of fixed-effect models. In place of this, we take $E(\beta_i^*) = 0$. Also, y_{ij}'s are no longer independently distributed but

$$Var(y_{ij}) = \sigma_\beta^2 + \sigma^2,$$

$$Cov(y_{ij}, y_{i'j'}) = \begin{cases} \sigma_\beta^2 & if \;\; i = i', j \neq j' \\ 0 & \text{otherwise.} \end{cases}$$

In the case of interblock analysis, we work with the block totals B_i in place of y_{ij} where

$$B_i = \sum_{j=1}^{v} n_{ij} y_{ij}$$
$$= \sum_{j=1}^{v} n_{ij} (\mu* + \beta_i^* + \tau_j + \varepsilon_{ij})$$
$$= k\mu* + \sum_j n_{ij}\tau_j + f_i$$

where $f_i = \beta_i^* k + \sum_j n_{ij}\varepsilon_{ij},\ (i = 1, 2, ..., b)$ are independent and normally distributed with mean 0 and

$$Var(f_i) = k^2\sigma_\beta^2 + k\sigma^2 = \sigma_f^2 .$$

Thus

$$E(B_i) = k\mu* + \sum_j n_{ij}\tau_j ,$$
$$Var(B_i) = \sigma_f^2; \quad i = 1, 2, .., b,$$
$$Cov(B_i, B_{i'}) = 0;\ i \neq i'; i, i' = 1, 2, ..., b.$$

In matrix notations, the model under consideration can be written as

$$B = k\mu * E_{b1} + N\tau + f$$

where $f = (f_1, f_2, ..., f_b)'$.

Estimates of $\mu*$ and τ in interblock analysis:

In order to obtain the estimates of $\mu*$ and τ, we minimise the sum of squares due to error $f = (f_1, f_2, ..., f_b)'$, , i.e., minimize $(B - k\mu * E_{b1} - N\tau)'(B - k\mu * E_{b1} - N\tau)$ with respect to $\mu*$ and τ .

The estimates of $\mu*$ and τ are the solutions of following normal equations:

$$\begin{pmatrix} kE_{b1}' \\ N' \end{pmatrix} \begin{pmatrix} kE_{b1} & N \end{pmatrix} \begin{pmatrix} \tilde{\mu} \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} kE_{b1}' \\ N' \end{pmatrix} B$$

or $$\begin{pmatrix} k^2 E_{b1}' E_{b1} & kE_{b1}' N \\ kN' E_{b1} & N'N \end{pmatrix} \begin{pmatrix} \tilde{\mu} \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} kG \\ N'B \end{pmatrix}$$

or $$\begin{pmatrix} k^2 b & kE_{v1}' R \\ kRE_{v1} & N'N \end{pmatrix} \begin{pmatrix} \tilde{\mu} \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} kG \\ N'B \end{pmatrix}$$ (using $N' E_{b1} = r = RE_{v1}$)

Premultiplying both sides of the equation by

$$\begin{pmatrix} 1 & 0 \\ -\frac{RE_{v1}}{b} & I_v \end{pmatrix},$$

we get

$$\begin{pmatrix} bk & E_{v1}^{'} \\ 0 & N'N - \frac{RE_{v1}E_{v1}^{'}R}{b} \end{pmatrix}\begin{pmatrix} \tilde{\mu} \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} G \\ N'B - \frac{RE_{v1}G}{b} \end{pmatrix}.$$

Using the side condition $E_{v1}^{'}R\tau = 0$ and assuming $N'N$ to be nonsingular, we get the estimates of μ^* and τ as $\tilde{\mu}$ and $\tilde{\tau}$ given by

$$\begin{aligned}
\tilde{\mu} &= \frac{G}{bk}, \\
\tilde{\tau} &= (N'N)^{-1}(N'B - \frac{RE_{v1}G}{b}) \\
&= (N'N)^{-1}\left(N'B - \frac{kGN'E_{b1}}{bk}\right) \quad (\text{using } RE_{v1} = r = N'E_{v1}) \\
&= (N'N)^{-1}\left(N'B - \frac{G}{bk}N'NE_{v1}\right) \\
&= (N'N)^{-1}N'B - \frac{GE_{v1}}{bk}.
\end{aligned}$$

The normal equations can also be solved in an alternative way also as follows.

The normal equations $\begin{pmatrix} k^2b & kE_{v1}^{'}R \\ kRE_{v1} & N'N \end{pmatrix}\begin{pmatrix} \tilde{\mu} \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} kG \\ N'\beta \end{pmatrix}$ can be written as

$$\begin{aligned}
k^2b\tilde{\mu} + kE_{v1}^{'}R\tilde{\tau} &= kG \\
kRE_{v1}\tilde{\mu} + N'N\tilde{\tau} &= N'B.
\end{aligned}$$

Using the side condition $E_{v1}^{'}R\tilde{\tau} = 0$ (or equivalently $\sum_j r_j\tilde{\tau}_j = 0$) and assuming $N'N$ to be nonsingular, the first equation gives $\tilde{\mu} = \frac{G}{bk}$. Substituting $\tilde{\mu}$ in the second equation gives $\tilde{\tau}$.

$$\begin{aligned}
\tilde{\tau} &= \left(N'N\right)^{-1}\left(N'B - \frac{RE_{v1}G}{b}\right) \\
&= \left(N'N\right)^{-1}N'B - \frac{GE_{v1}}{bk}.
\end{aligned}$$

Generally, we are not interested merely in the interblock analysis of variance but we want to utilize the information from interblock analysis along with the intrablock information to improve upon the statistical inferences.

After obtaining the interblock estimate of treatment effects, the next question that arises is how to use this information for improved estimation of treatment effects and use it further for the testing of significance of treatment effects. Such an estimate will be based on the use of more information, so it is expected to provide better statistical inferences.

We now have two different estimates of the treatment effect as

- based on intrablock analysis $\hat{\tau} = C^{-}Q$ and
- based on interblock analysis $\tilde{\tau} = (N'N)^{-1}N'B - \frac{GE_{v1}}{bk}$.

Let us consider the estimation of linear contrast of treatment effects $L = l'\tau$. Since the intrablock and interblock estimates of τ are based on Gauss-Markov model and least-squares principle, so the best estimate of L based on intrablock estimation is

$$\begin{aligned} L_1 &= l'\hat{\tau} \\ &= l'C^{-}Q \end{aligned}$$

and the best estimate of L based on interblock estimation is

$$\begin{aligned} L_2 &= l'\tilde{\tau} \\ &= l'\left[(N'N)^{-1}N'B - \frac{GE_{v1}}{bk}\right] \\ &= l'(N'N)^{-1}N'B \quad \text{(since } l'E_{v1} = 0 \text{ being contrast.)} \end{aligned}$$

The variances of L_1 and L_2 are

$$Var(L_1) = \sigma^2 l'C^{-}l$$

and

$$Var(L_2) = \sigma_f^2 l'(N'N)^{-1}l,$$

respectively. The covariance between Q (from intrablock) and B (from interblock) is

$$\begin{aligned} Cov(Q,B) &= Cov(V - N'K^{-1}B^*, B) \\ &= Cov(V,B) - Cov(N'K^{-1}B^*, B) \\ &= N'\sigma_f^2 - N'K^{-1}K\sigma_f^2 \\ &= 0. \end{aligned}$$

Note that B^* denotes the block total based on intrablock analysis and B denotes the block totals based on interblock analysis. We are using two notations B and B^* just to indicate that the two block totals are different. The reader should not misunderstand that it follows from the result $Cov(Q, B) = 0$ in case of intrablock analysis.

Thus

$$Cov(L_1, L_2) = 0$$

irrespective of the values of l.

The question now arises that given the two estimators $\hat{\tau}$ and $\tilde{\tau}$ of τ, how to combine them and obtain a minimum variance unbiased estimator of τ. It is illustrated with the following example:

Example:

Let $\hat{\varphi}_1$ and $\hat{\varphi}_2$ be any two unbiased estimators of a parameter φ with $Var(\hat{\varphi}_1) = \sigma_1^2$ and $Var(\hat{\varphi}_2) = \sigma_2^2$.

Consider a linear combination $\hat{\varphi} = \theta_1\hat{\varphi}_1 + \theta_2\hat{\varphi}_2$ with weights θ_1 and θ_2. In order that $\hat{\varphi}$ is an unbiased estimator of θ_1, we need

$$E(\hat{\varphi}) = \varphi$$

or $\quad \theta_1 E(\hat{\varphi}_1) + \theta_2 E(\hat{\varphi}_2) = \varphi$

or $\quad \theta_1\varphi + \theta_2\varphi = \varphi$

or $\quad \theta_1 + \theta_2 = 1.$

So modify $\hat{\varphi}$ as $\dfrac{\theta_1\hat{\varphi}_1 + \theta_2\hat{\varphi}_2}{\theta_1 + \theta_2}$ which is the weighted mean of $\hat{\varphi}_1$ and $\hat{\varphi}_2$.

Further, if $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are independent, then

$$Var(\hat{\varphi}) = \theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2.$$

Now we find θ_1 and θ_2 such that $Var(\hat{\varphi})$ is minimum such that $\theta_1 + \theta_2 = 1$.

$$\frac{\partial Var(\hat{\varphi})}{\partial \theta_1} = 0 \Rightarrow 2\theta_1\sigma_1^2 - 2(1-\theta_1)\sigma_2^2 = 0$$

or $\quad \theta_1\sigma_1^2 - \theta_2\sigma_2^2 = 0$

or $\quad \dfrac{\theta_1}{\theta_2} = \dfrac{\sigma_2^2}{\sigma_1^2}$

or $\quad \text{weight} \propto \dfrac{1}{\text{variance}}.$

Alternatively, the Lagrangian function approach can be used to obtain such a result as follows. The Lagrangian function with λ^* as Lagrangian multiplier is given by

$$\phi = Var(\hat{\varphi}) - \lambda^*(\theta_1 + \theta_2 - 1)$$

and solving $\frac{\partial \phi}{\partial \theta_1} = 0, \frac{\partial \phi}{\partial \theta_2}$ and $\frac{\partial \phi}{\partial \lambda^*} = 0$ also gives the same result that $\frac{\theta_1}{\theta_2} = \frac{\sigma_2^2}{\sigma_1^2}$.

We note that a pooled estimator of τ in the form of weighted arithmetic mean of uncorrelated L_1 and L_2 is the minimum variance unbiased estimator of τ when the weights θ_1 and θ_2 of L_1 and L_2, respectively are chosen such that

$$\frac{\theta_1}{\theta_2} = \frac{Var(L_2)}{Var(L_1)},$$

i.e., the chosen weights are reciprocal to the variance of respective estimators, irrespective of the values of l. So consider the weighted average of L_1 and L_2 with weights θ_1 and θ_2, respectively as

$$\begin{aligned}\tau^* &= \frac{\theta_1 L_1 + \theta_2 L_2}{\theta_1 + \theta_2} \\ &= \frac{l'(\theta_1 \hat{\tau} + \theta_2 \tilde{\tau})}{\theta_1 + \theta_2}\end{aligned}$$

with

$$\theta_1^{-1} = l'C^{-1}l\sigma^2$$
$$\theta_2^{-1} = l'(N'N)^{-1}l\sigma_f^2.$$

The linear contrast of τ^* is

$$L^* = l'\tau^*$$

and its variance is

$$\begin{aligned}Var(L^*) &= \frac{\theta_1^2 Var(L_1) + \theta_2^2 Var(L_2)}{(\theta_1 + \theta_2)^2} l'l \quad (\text{since } Cov(L_1, L_2) = 0) \\ &= \frac{l'l}{(\theta_1 + \theta_2)}\end{aligned}$$

because the weights of estimators are chosen to be inversely proportional to the variance of the respective estimators. We note that τ^* can be obtained provided θ_1 and θ_2 are known. But θ_1 and θ_2 are known only if σ^2 and σ_β^2 are known. So τ^* can be obtained when σ^2 and σ_β^2 are known. In case, if σ^2 and σ_β^2 are unknown, then their estimates can be used. A question arises how to obtain such estimators? One such approach to obtain the estimates of σ^2 and σ_β^2 is based on utilizing the results from intrablock and interblock analysis both and is as follows.

From intrablock analysis

$$E(SS_{Error(t)}) = (n-b-v+1)\sigma^2,$$

so an unbiased estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{SS_{Error(t)}}{n-b-v+1}.$$

An unbiased estimator of σ_β^2 is obtained by using the following results based on the intrablock analysis:

$$SS_{Treat(unadj)} = \sum_{j=1}^{v} \frac{V_j^2}{\tau_j} - \frac{G^2}{n},$$

$$SS_{Block(unadj)} = \sum_{i=1}^{b} \frac{B_i^2}{k_i} - \frac{G^2}{n},$$

$$SS_{Treat(adj)} = \sum_{j=1}^{v} Q_j \hat{\tau}_j,$$

$$SS_{Total} = \sum_{i=1}^{b}\sum_{j=1}^{v} y_{ij}^2 - \frac{G^2}{n},$$

where

$$\begin{aligned} SS_{Total} &= SS_{Treat(adj)} + SS_{Block(unadj)} + SS_{Error(t)} \\ &= SS_{Treat(unadj)} + SS_{Block(adj)} + SS_{Error(t)}. \end{aligned}$$

Hence

$$SS_{Block(adj)} = SS_{Treat(adj)} + SS_{Block(unadj)} - SS_{Treat(unadj)}.$$

Under the interblock analysis model

$$E[SS_{Block(adj)}] = E[SS_{Treat(adj)}] + E[SS_{Block(unadj)}] - E[SS_{Treat(unadj)}]$$

which is obtained as follows:

$$E[SS_{Block(adj)}] = (b-1)\sigma^2 + (n-v)\sigma_\beta^2$$

or

$$E\left[SS_{Block(adj)} - \frac{b-1}{n-b-v+1} SS_{Error(t)}\right] = (n-v)\sigma_\beta^2.$$

Thus an unbiased estimator of σ_β^2 is

$$\hat{\sigma}_\beta^2 = \frac{1}{n-v}\left[SS_{Block(adj)} - \frac{b-1}{n-b-v+1} SS_{Error(t)}\right].$$

Now the estimates of weights θ_1 and θ_2 can be obtained by replacing σ^2 and σ_β^2 by $\hat{\sigma}^2$ and $\hat{\sigma}_\beta^2$ respectively. Then the estimate of $\tau*$ can be obtained by replacing θ_1 and θ_2 by their estimates and can be used in place of $\tau*$. It may be noted that the exact distribution of the associated sum of squares due to treatments is difficult to find when
σ^2 and σ_β^2 are replaced by $\hat{\sigma}^2$ and $\hat{\sigma}_\beta^2$, respectively in $\tau*$. Some approximate results are possible which we will present while dealing with the balanced incomplete block design. An increase in the precision using interblock analysis as compared to intrablock analysis is measured by

$$\frac{1/\text{variance of pooled estimate}}{1/\text{variance of intrablock estimate}} - 1.$$

In the interblock analysis, the block effects are treated as a random variable which is appropriate if the blocks can be regarded as a random sample from a large population of blocks. The best estimate of the treatment effect from the intrablock analysis is further improved by utilizing the information on block totals. Since the treatments in different blocks are not all the same, so the difference between block totals is expected to provide some information about the differences between the treatments. So the interblock estimates are obtained and pooled with intrablock estimates to obtain the combined estimate of τ. The procedure of obtaining the interblock estimates and then obtaining the pooled estimates is called the **recovery of interblock information**.