Chapter 6

Balanced Incomplete Block Design (BIBD)

The designs like CRD and RBD are the complete block designs. We now discuss the balanced incomplete block design (BIBD) and the partially balanced incomplete block design (PBIBD) which are the incomplete block designs.

A balanced incomplete block design (BIBD) is an incomplete block design in which

- *b* blocks have the same number *k* of plots each and
- every treatment is replicated *r* times in the design.
- Each treatment occurs at most once in a block, i.e., $n_{ij} = 0$ or 1 where n_{ij} is the number of times the j^{th} treatment occurs in i^{th} block, $i = 1, 2, ..., b; j = 1, 2, ..., v.$
- Every pair of treatments occurs together is λ of the *b* blocks.

Five parameters denote such design as $D(b, k, v, r; \lambda)$.

The parameters b, k, v, r and λ are not chosen arbitrarily.

They satisfy the following relations:

(I) $bk = vr$

(II) $\lambda(v-1) = r(k-1)$

(III) $b \geq v$ (and hence $r > k$).

$$\text{Hence } \sum_j n_{ij} = k \qquad \text{for all } i$$

$$\sum_j n_{ij} = r \qquad \text{for all } j$$

and $n_{1j}n_{ij'} + n_{2j}n_{ij'} + ... + n_{b_j}n_{b_{j'}} = \lambda$ for all $j \neq j' = 1, 2, ..., v.$ Obviously $\dfrac{n_{ij}}{r}$ cannot be a constant for all j. So the design is not orthogonal.

Example of BIBD

In the design $D(b,k;v,r;\lambda)$: consider $b=10$ $(say, B_1,...,B_{10})$, $v=6$ $(say, T_1,...,T_6)$, $k=3, r=5, \lambda=2$

Blocks	Treatments
B_1	T_1 T_2 T_5
B_2	T_1 T_2 T_6
B_3	T_1 T_3 T_4
B_4	T_1 T_3 T_6
B_5	T_1 T_4 T_5
B_6	T_2 T_3 T_4
B_7	T_2 T_3 T_5
B_8	T_2 T_4 T_6
B_9	T_3 T_5 T_6
B_{10}	T_4 T_5 T_6

Now we see how the conditions of BIBD are satisfied.

(i) $bk = 10\times 3 = 30$ and $vr = 6\times 5 = 30$

$\Rightarrow bk = vr$

(ii) $\lambda(v-1) = 2\times 5 = 10$ and $r(k-1) = 5\times 2 = 10$

$\Rightarrow \lambda(v-1) = r(k-1)$

(iii) $b = 10 \geq 6$

Even if the parameters satisfy the relations, it is not always possible to arrange the treatments in blocks to get the corresponding design.

The necessary and sufficient conditions to be satisfied by the parameters for the existence of a BIBD are not known.

The conditions (I)-(III) are some necessary condition only. The construction of such design depends on the actual arrangement of the treatments into blocks and this problem is handled in combinatorial mathematics. Tables are available, giving all the designs involving at most 20 replication and their method of construction.

Theorem:

$(I) bk = vr$

$(II)\, \lambda(v-1) = r(k-1)$

$(III)\, b \geq v.$

Proof: (I)

Let $N = (n_{ij}) : b \times v$ the incidence matrix

Observing that the quantities $E_{1b}NE_{v1}$ and $E_{1v}N'E_{b1}$ are the scalars and the transpose of each other, we find their values.

Consider

$$E_{1b}NE_{v1} = (1,1,...,1)\begin{pmatrix} n_{11} & n_{21} \cdots n_{b1} \\ n_{12} & n_{22} \cdots n_{b2} \\ \vdots & \vdots \ \ddots \ \vdots \\ n_{1v} & n_{2v} \cdots n_{bv} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= (1,1,...,1)\begin{pmatrix} \sum_j n_{1j} \\ \sum_j n_{2j} \\ \vdots \\ \sum_j n_{bj} \end{pmatrix}$$

$$= (1,1,...,1)_{1\times b}\begin{pmatrix} k \\ k \\ \vdots \\ k \end{pmatrix} = bk.$$

Similarly,

$$E_{1v}N'E_{b1} = (1,...,1)\begin{pmatrix} n_{11} & n_{21} \cdots n_{b1} \\ n_{12} & n_{22} \cdots n_{b2} \\ \vdots & \vdots \ \ddots \ \vdots \\ n_{1v} & n_{2v} \cdots n_{bv} \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

$$=(1,1,...,1)\begin{pmatrix} \sum_i n_i \\ \vdots \\ \sum_i n_{iv} \end{pmatrix} = (1,1,...,1)_{1\times v}\begin{pmatrix} r \\ \vdots \\ r \end{pmatrix} = vr$$

But

$E_{1b}NE_{v1} = E_{1v}N'E_{b1}$ as both are scalars.

Thus $bk = vr.$

Proof: (II)

Consider

$$N'N = \begin{pmatrix} n_{11} & n_{21} \cdots n_{b1} \\ n_{12} & n_{22} \cdots n_{b2} \\ \vdots & \vdots \ \ddots \ \vdots \\ n_{1v} & n_{2v} \cdots n_{bv} \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \cdots n_{1v} \\ n_{21} & n_{22} \cdots n_{2v} \\ \vdots & \vdots \ \ddots \ \vdots \\ n_{b1} & n_{b2} \cdots n_{bv} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_i n_{i1}^2 & \sum_i n_{i1}n_{i2} \cdots \sum_i n_{i1}n_{iv} \\ \sum_i n_{i1}n_{i2} & \sum_i n_{i2}^2 \cdots \sum_i n_{i2}n_{iv} \\ \vdots & \vdots \quad \ddots \quad \vdots \\ \sum_i n_{iv}n_{i1} & \sum_i n_{iv}n_{i2} \cdots \sum_i n_{iv}^2 \end{pmatrix}$$

$$= \begin{pmatrix} r & \lambda & \cdots \lambda \\ \lambda & r & \cdots \lambda \\ \vdots & \vdots & \vdots \\ r & \lambda & \cdots \lambda \end{pmatrix}. \qquad (1)$$

Since $n_{ij}^2 = 1$ or 0 as $n_{ij} = 1$ or 0,

so $\sum_i n_{ij}^2$ = Number of times τ_j occurs in the design

$= r$ for all $j = 1, 2, ..., v$ of times occurs in the design

and $\sum_i n_{ij}n_{ij'}$ = Number of blocks in which τ_j and $\tau_{j'}$ occurs together

$= \lambda$ for all $j \neq j'$.

$$N'NE_{v1} = \begin{pmatrix} r & \lambda \cdots \lambda \\ \lambda & r \cdots \lambda \\ \vdots & \vdots \ddots \vdots \\ \lambda & \lambda \cdots r \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} r + \lambda(v-1) \\ r + \lambda(v-1) \\ \vdots \\ r + \lambda(v-1) \end{pmatrix} = [r + \lambda(v-1)]E_{v1}. \qquad (2)$$

Also

$$N'NE_{v1} = N'\left[\begin{pmatrix} n_{11} & n_{12} \cdots n_{1v} \\ n_{21} & n_{22} \cdots n_{2v} \\ \vdots & \vdots \ \ddots \ \vdots \\ n_{b1} & n_{b2} \cdots n_{bv} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right]$$

$$= N'\begin{pmatrix} \sum_j n_{1j} \\ \sum_j n_{2j} \\ \vdots \\ \sum_j n_{bj} \end{pmatrix}$$

$$= \begin{pmatrix} n_{11} & n_{21} \cdots n_{b1} \\ n_{12} & n_{22} \cdots n_{b2} \\ \vdots & \\ n_{iv} & n_{2v} \cdots n_{bv} \end{pmatrix}\begin{pmatrix} k \\ k \\ \vdots \\ k \end{pmatrix}_{b\times 1}$$

$$= k\begin{pmatrix} \sum_i n_{i1} \\ \sum_i n_{i2} \\ \vdots \\ \sum_i n_{iv} \end{pmatrix}$$

$$= k\begin{pmatrix} r \\ r \\ \vdots \\ r \end{pmatrix}$$

$$= krE_{v1} \qquad (3)$$

From (2) and (3)

$$[r+\lambda(v-1)]E_{v1} = krE_{v1}$$

or $\quad r+\lambda(v-1) = kr$

or $\quad \lambda(v-1) = r(k-1)$

Proof: (III)

From (I), the determinant of $N'N$ is

$$\begin{aligned} \det\left|N'N\right| &= [r+\lambda(v-1)](r-\lambda)^{v-1} \\ &= \left[r+r(k-1)\right]\left(r-\lambda\right)^{v-1} \\ &= rk(r-\lambda)^{v-1} \\ &\neq 0 \end{aligned}$$

because since if $r=\lambda \Rightarrow$ from (II) that $k = v$. This contradicts the incompleteness of the design.

Thus $N'N$ is a $v \times v$ nonsingular matrix.

Thus $rank(N'N) = v.$

We know from matrix theory result

$rank(N) = rank(N'N)$

so $rank(N) = v$

But $rank(N) \leq b$, there being b rows in $N.$

Thus $v \leq b.$

Interpretation of conditions of BIBD

Interpretation of (I) $bk = vr$

This condition is related to the total number of plots in an experiment. In our settings, there are k plots in each block and there are b blocks. So the total number of plots are bk.

Further, there are v treatments and each treatment is replicated r times such that each treatment occurs atmost in one block. So total number of plots containing all the treatments is vr. Since both the statements counts the total number of plots, hence $bk = vr.$

Interpretation of (II)

Each block has k plots. Thus the total pairs of plots in a block = $\binom{k}{2} = \frac{k(k-1)}{2}.$

There are b blocks. Thus the total pairs of plots such that each pair consists of plots within a block = $b\frac{k(k-1)}{2}$.

There are v treatments, thus the total number of pairs of treatment = $\binom{v}{2} = \frac{v(v-1)}{2}$.

Each pair of treatment is replicated λ times, i.e., each pair of treatment occurs in λ blocks.

Thus the total number of pairs of plots within blocks must be $= \lambda\frac{v(v-1)}{2}$.

Hence $b\frac{k(k-1)}{2} = \lambda\frac{v(v-1)}{2}$

Using $bk = vr$ in this relation, we get $r(k-1) = \lambda(v-1).$

Proof of (III) was given by Fisher but quite long, so not needed here.

Balancing in designs:

There are two types of balancing – Variance balanced and efficiency balanced. We discuss the variance balancing now and the efficiency balancing later.

Balanced Design (Variance Balanced):

A connected design is said to be balanced (variance balanced) if all the elementary contrasts of the treatment effects can be estimated with the same precision. This definition does not hold for the disconnected design, as all the elementary contrasts are not estimable in this design.

Proper Design:

An incomplete block design with $k_1 = k_2 = ... = k_b = k$ is called a proper design.

Symmetric BIBD:

A BIBD is called symmetrical if the number of blocks = number of treatments, i.e., $b = v.$

Since $b = v,$ so from $bk = vr$

$$\Rightarrow k = r.$$

Thus the number of pairs of treatments common between any two blocks = λ .

The determinant of $N'N$ is

$$\begin{aligned}|N'N| &= [r + \lambda(v-1)](r-\lambda)^{v-1}\\ &= \left[r + r(k-1)\right]\left(r-\lambda\right)^{v-1}\\ &= rk(r-\lambda)^{v-1}.\end{aligned}$$

When BIBD is symmetric, $b = v$ and then using $bk = vr,$ we have $k = r.$ Thus

$$|N'N| = |N|^2 = r^2(r-\lambda)^{v-1},$$

so

$$|N| = \pm\, r(r-\lambda)^{\frac{v-1}{2}}.$$

Since $|N|$ is an integer, hence when v is an even number, $(r-\lambda)$ must be a perfect square. So

$$\begin{aligned}|N'N| &= (r-\lambda)I + \lambda E_{v1}E_{v1}',\\ (N'N)^{-1} &= N^{-1}N'^{-1}\\ &= \frac{1}{r-\lambda}\left[I - \frac{\lambda}{r^2}E_{v1}E_{v1}'\right],\\ N'^{-1} &= \frac{1}{r-\lambda}\left[I - \frac{\lambda}{r^2}E_{v1}E_{v1}'\right].\end{aligned}$$

Post-multiplying both sides by N', we get

$$NN' = (r-\lambda)I + \lambda E_{v1}E_{v1}' = N'N.$$

Hence in the case of a symmetric BIBD, any two blocks have λ treatment in common.

Since BIBD is an incomplete block design. So every pair of treatment can occur at most once is a block, we must have $v \geq k.$

If $v = k,$ then it means that each treatment occurs once in every block which occurs in case of RBD. So in BIBD, always assume $v > k$.

Similarly $\lambda < r.$

[If $\lambda = r$ then $\lambda(v-1) = r(k-1) \Rightarrow v = k \Rightarrow$ which means that the design is RBD]

Resolvable design:

A block design of

- b blocks in which
- each of v treatments is replicated r times

is said to be resolvable if b blocks can be divided into r sets of b/r blocks each, such that every treatment appears in each set precisely once. Obviously, in a resolvable design, b is a multiple of r.

Theorem: If in a BIBD $D(v,b,r,k,\lambda),$ b is divisible by r, then

$$b \geq v + r - 1.$$

Proof: Let $b = nr$ (where $n > 1$ is a positive integer).

For a BIBD, $\lambda(v-1) = r(k-1)$

$$\text{or} \quad r = \frac{\lambda(v-1)}{(k-1)} \qquad \begin{bmatrix} \text{because } vr = bk \\ \text{or} \quad vr = nrk \\ \text{or} \quad v = nk \end{bmatrix}$$

$$= \frac{\lambda(nk-1)}{(k-1)}$$

$$= \lambda\left(\frac{n-1}{k-1}\right) + \lambda n.$$

Since $n > 1$ and $k > 1$, so $\lambda n > 1$ is an integer. Since r has to be an integer.

$$\Rightarrow \lambda\frac{(n-1)}{k-1} \text{ is also a positive integer.}$$

Now, if possible, let

$$b < v + r - 1$$
$$\Rightarrow nr < v + r - 1$$
$$\text{or } r(n-1) < v - 1$$
$$\text{or } r(n-1) < \frac{r(k-1)}{\lambda} \qquad \left(\text{because } v - 1 = \frac{r(k-1)}{\lambda}\right)$$
$$\Rightarrow \frac{\lambda(n-1)}{k-1} < 1 \text{ which is a contradiction as integer can not be less than one}$$

$\Rightarrow b < v + r - 1$ is impossible. Thus the opposite is true.

$\Rightarrow b \geq v + r - 1$ holds correct.

Intrablock analysis of BIBD:

Consider the model

$$y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij};\ i = 1, 2, ..., b;\ j = 1, 2, ..., v,$$

where

μ is the general mean effect;

β_i is the fixed additive i^{th} block effect;

τ_j is the fixed additive j^{th} treatment effect and

ε_{ij} is the i.i.d. random error with $\varepsilon_{ij} \sim N(0, \sigma^2)$.

We don't need to develop the analysis of BIBD from starting. Since BIBD is also an incomplete block design and the analysis of incomplete block design has already been presented in the earlier module, so we implement those derived expressions directly under the setup and conditions of BIBD. Using the same notations, we represent the blocks totals by $B_i = \sum_{j=1}^{v} y_{ij}$, treatment totals by $V_j = \sum_{i=1}^{b} y_{ij}$, adjusted treatment totals by Q_j and grand total by $G = \sum_{i=1}^{b} \sum_{j=}^{v} y_{ij}$ The normal equations are obtained by differentiating the error sum of squares. Then the block effects are eliminated from the normal equations and the normal equations are solved for the treatment effects. The resulting intrablock equations of treatment effects in matrix notations are expressible as

$$Q = C\hat{\tau}.$$

Now we obtain the forms of C and Q in the case of BIBD. The diagonal elements of C are given by

$$c_{jj} = r - \frac{\sum_{i=1}^{b} n_{ij}^2}{k} \quad (j = 1, 2, ..., v)$$
$$= r - \frac{r}{k}.$$

The off-diagonal elements of C are given by

$$\begin{aligned} c_{jj'} &= -\frac{1}{k}\sum_{i=1}^{b} n_{ij}n_{ij'} \quad (j \neq j'; j, j' = 1, 2, ..., v) \\ &= -\frac{\lambda}{k}. \end{aligned}$$

The adjusted treatment totals are obtained as

$$\begin{aligned} Q_j &= V_j - \frac{1}{k}\sum_{i=1}^{b} n_{ij}B_i \quad (j \neq 1, 2, ..., v) \\ &= V_j - \frac{1}{k}\sum_{i(j)} B_i \end{aligned}$$

where $\sum_{i(j)}$ denotes the sum over those blocks containing j^{th} treatment. Denote

$T_j = \sum_{i(j)} B_i$, then

$Q_j = V_j - \frac{T_j}{k}.$

The C matrix is simplified as follows:

$$\begin{aligned} C &= rI - \frac{N'N}{k} \\ &= rI - \frac{1}{k}\left[(r-\lambda)I + \lambda E_{v1}E_{v1}'\right] \\ &= r\left(\frac{k-1}{k}\right)I + \frac{\lambda}{k}(I - E_{v1}E_{vi}') \\ &= \lambda\left(\frac{v-1}{k}\right)I + \frac{\lambda}{k}(I - E_{v1}E_{vi}') \\ &= \frac{\lambda V}{k}\left(I - \frac{E_{v1}E_{v1}'}{v}\right). \end{aligned}$$

Since C is not as a full rank matrix, so its unique inverse does not exist. The generalized inverse of C is denoted as C^- which is obtained as

$$C^- = \left(C + \frac{E_{v1}E_{v1}'}{v}\right)^{-1}.$$

Since

$$C = \frac{\lambda v}{k}\left(I_v - \frac{E_{v1}E_{v1}'}{v}\right)$$

$$\text{or } \frac{kC}{\lambda v} = I_v - \frac{E_{v1}E_{v1}'}{v},$$

the generalized inverse of $\frac{k}{\lambda v}C$ is

$$\begin{aligned}\left(\frac{k}{\lambda v}\right)^{-1}C^{-} &= \left[C + \frac{E_{v1}E_{v1}^{'}}{v},\right]^{-1} \\ &= \left[I_v - \frac{E_{v1}E_{v1}^{'}}{v} + \frac{E_{v1}E_{v1}^{'}}{v}\right]^{-1} \\ &= I_v.\end{aligned}$$

Thus $C^{-} = \frac{\lambda v}{k} I_v$.

Thus an estimate of τ is obtained from $Q = C\tau$ as

$$\begin{aligned}\hat{\tau} &= C^{-}Q \\ &= \frac{\lambda v}{k}Q.\end{aligned}$$

The null hypothesis of our interest is $H_0 : \tau_1 = \tau_2 = ... = \tau_v$ against the alternative hypothesis H_1 : at least one pair of τ_j 's is different. Now we obtain the various sum of squares involved in the development of analysis of variance as follows.

The adjusted treatment sum of squares is

$$\begin{aligned}SS_{Treat(adj)} &= \hat{\tau}'Q \\ &= \frac{k}{\lambda v}Q'Q \\ &= \frac{k}{\lambda v}\sum_{j=1}^{v} Q_j^2,\end{aligned}$$

The unadjusted block sum of squares is

$$SS_{Block(unadj)} = \sum_{i=1}^{b}\frac{B_i^2}{k} - \frac{G^2}{bk}.$$

The total sum of squares is

$$SS_{Total} = \sum_{i=1}^{b}\sum_{j=1}^{v} y_{ij}^2 - \frac{G^2}{bk}$$

The residual sum of squares is obtained by

$$SS_{Error(t)} = SS_{Total} - SS_{Block(unadj)} - SS_{Treat(adj)}.$$

A test for $H_0 : \tau_1 = \tau_2 = ... = \tau_v$ is then based on the statistic

$$F_{Tr} = \frac{SS_{Treat(adj)} / (v-1)}{SS_{Error(t)} / (bk - b - v + 1)}$$

$$= \frac{k}{\lambda v} . \frac{bk - b - v + 1}{v - 1} . \frac{\sum_{j=1}^{v} Q_j^2}{SS_{Error(t)}}$$

If $F_{Tr} > F_{1-\alpha, v-1, bk-b-v+1;}$ then $H_{0(t)}$ is rejected.

This completes the analysis of variance test and is termed as intrablock analysis of variance. This analysis can be compiled into the intrablock analysis of variance table for testing the significance of the treatment effect given as follows.

Intrablock analysis of variance table of BIBD for $H_0 : \tau_1 = \tau_2 = ... = \tau_v$

Source	Sum of squares	Degrees of freedom	Mean squares	F
Between treatment (adjusted)	$SS_{Treat(adj)}$	$v-1$	$MS_{treat} = \frac{SS_{Treat(adj)}}{v-1}$	$\frac{MS_{Treat}}{MS_E}$
Between blocks (unadjusted)	$SS_{Block(unadj)}$	b - 1		
Intrablock error	$SS_{Error(t)}$ (by substraction)	$bk - b - v + 1$	$MS_E = \frac{SS_{Error(t)}}{bk - b - v + 1}$	
Total	$SS_{Total} = \sum_i \sum_j y_{ij}^2 - \frac{G^2}{bk}$	$bk - 1$		

In case, the null hyperthesis is rejected, then we go for a pairwise comparison of the treatments. For that, we need an expression for the variance of the difference of two treatment effects.

The variance of an elementary contrast $(\tau_j - \tau_{j'}, j \neq j')$ under the intrablock analysis is

$$\begin{aligned}
V^* &= Var(\hat{\tau}_j - \hat{\tau}_{j'}) \\
&= Var\left(\frac{k}{\lambda v}(Q_j - Q_{j'})\right) \\
&= \frac{k^2}{\lambda^2 v^2}[Var(Q_j) + Var(Q_j) - 2Cov(Q_j Q_{j'})] \\
&= \frac{k^2}{\lambda^2 v^2}(c_{jj} + c_{j'j'} - 2c_{jj'})\sigma^2 \\
&= \frac{k^2}{\lambda^2 v^2}\left[2r\left(1 - \frac{1}{k}\right) + \frac{2\lambda}{k}\right]\sigma^2 \\
&= \frac{2k}{\lambda v}\sigma^2.
\end{aligned}$$

This expression depends on σ^2 which is unknown. So it is unfit for use in the real data applications. One solution is to estimate σ^2 from the given data and use it in the place of σ^2.

An unbiased estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{SS_{Error(t)}}{bk - b - v + 1}.$$

Thus an unbiased estimator of V^* can be obtained by substituting $\hat{\sigma}^2$ in it as

$$\hat{V}_* = \frac{2k}{\lambda v} \cdot \frac{SS_{Error(t)}}{bk - b - v + 1}.$$

If H_0 is rejected, then we make pairwise comparison and use the multiple comparison test. To test $H_0 : \tau_j = \tau_{j'} (j \neq j'),$ a suitable statistic is

$$t = \frac{k(bk - b - v + 1)}{\lambda v} \cdot \frac{Q_j - Q_{j'}}{\sqrt{SS_{Error(t)}}}$$

which follows a *t*-distribution with $(bk - b - v + 1)$ degrees of freedom under H_0.

A question arises that how a BIBD compares to an RBD. Note that BIBD is an incomplete block design whereas RBD is a complete block design. This point should be kept in mind while making such restrictive comparison.

We now compare the efficiency of BIBD with a randomized block (complete) design with r replicates. The variance of an elementary contrast under a randomized block design (RBD) is

$$V_R^* = Var(\hat{\tau}_j^2 - \hat{\tau}_{j'})_{RBD} = \frac{2\sigma_*^2}{r}$$

where $Var(y_{ij}) = \sigma_*^2$ under RBD.

Thus the relative efficiency of BIBD relative to RBD is

$$\frac{Var(\hat{\tau}_j - \hat{\tau}_{j'})_{RBD}}{Var(\hat{\tau}_j - \hat{\tau}_{j'})_{BIBD}} = \frac{\left(\dfrac{2\sigma_*^2}{r}\right)}{\left(\dfrac{2k\sigma^2}{\lambda v}\right)}$$

$$= \frac{\lambda v}{rk}\left(\frac{\sigma_*^2}{\sigma^2}\right).$$

The factor $\dfrac{\lambda v}{rk} = E$ (say) is termed as the **efficiency factor** of BIBD and

$$E = \frac{\lambda v}{rk} = \frac{v}{k}\left(\frac{k-1}{v-1}\right)$$

$$= \left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{v}\right)^{-1}$$

$$< 1 \text{ (since } v > k).$$

The actual efficiency of BIBD over RBD not only depends on the efficiency factor but also on the ratio of variances σ_*^2 / σ^2. So BIBD can be more efficient than RBD as σ_*^2 can be more than σ^2 because $k < v$.

Efficiency balanced design:

A block design is said to be efficiency balanced if every contrast of the treatment effects is estimated through the design with the same efficiency factor.

If a block design satisfies any two of the following properties:

(i) efficiency balanced,

(ii) variance balanced and

(iii) an equal number of replications,

then the third property also holds true.

Missing observations in BIBD:

The intrablock estimate of missing $(i, j)^{th}$ observation y_{ij} is

$$y_{ij} = \frac{vr(k-1)B_i - k(v-1)Q_j - (v-1)Q_j^{'}}{k(k-1)(bk-b-v+1)}$$

$Q_j^{'}$: the sum of Q value for all other treatment (but not the j^{th} one) which are present in the i^{th} block.

All other procedures remain the same.

Interblock analysis and recovery of interblock information in BIBD

In the intrablock analysis of variance of an incomplete block design or BIBD, the treatment effects were estimated after eliminating the block effects from the normal equations. In a way, the block effects were assumed to be not marked enough and so they were eliminated. It is possible in many situations that the block effects are influential and marked. In such situations, the block totals may carry information about the treatment combinations also. This information can be used in estimating the treatment effects which may provide more efficient results. This is accomplished by an interblock analysis of BIBD and used further through the recovery of interblock information. So we first conduct the interblock analysis of BIBD. We do not derive the expressions a fresh but we use the assumptions and results from the interblock analysis of an incomplete block design. We additionally assume that the block effects are random with variance σ_β^2.

After estimating the treatment effects under interblock analysis, we use the results for the pooled estimation and recovery of interblock information in a BIBD.

In case of BIBD,

$$
\begin{aligned}
N'N &= \begin{pmatrix} \sum_i n_{i1}^2 & \sum_i n_{i1}n_{i2} & \cdots \sum_i n_{i1}n_{iv} \\ \sum_i n_{i1}n_{i2} & \sum_i n_{i2}^2 & \cdots & \sum_i n_{i2}n_{iv} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_i n_{iv}n_{i1} & \sum_i n_{iv}n_{i2} & \cdots & \sum_i n_{iv}^2 \end{pmatrix} \\
&= \begin{pmatrix} r & \lambda \cdots \lambda \\ \lambda & r \cdots \lambda \\ \vdots & \vdots \ddots \vdots \\ \lambda & \lambda \cdots r \end{pmatrix} \\
&= (r-\lambda)I_v + \lambda E_{v1}E_{v1}^{'}
\end{aligned}
$$

$$
(N'N)^{-1} = \frac{1}{r-\lambda}\left[I_v - \frac{\lambda E_{v1}E_{v1}^{'}}{rk} \right]
$$

The interblock estimate of τ can be obtained by substituting the expression on $\left(N'N\right)^{-1}$ in the earlier obtained interblock estimate.

$$
\tilde{\tau} = (N'N)^{-1}N'B - \frac{GE_{v1}}{bk}.
$$

Our next objective is to use the intrablock and interblock estimates of treatment effects together to find an improved estimate of treatment effects.

In order to use the interblock and intrablock estimates of τ together through pooled estimate, we consider the interblock and intrablock estimates of the treatment contrast.

The intrablock estimate of treatment contrast $l'\tau$ is

$$\begin{aligned} l'\hat{\tau} &= l'C^{-}Q \\ &= \frac{k}{\lambda v} l'Q \\ &= \frac{k}{\lambda v} \sum_j l_j Q_j \\ &= \sum_j l_j \hat{\tau}_j, \text{ say.} \end{aligned}$$

The interblock estimate of treatment contrast $l'\tau$ is

$$\begin{aligned} l'\tilde{\tau} &= \frac{l'N'B}{r-\lambda} \quad (\text{since } l'E_{v1} = 0) \\ &= \frac{1}{r-\lambda} \sum_{j=1}^{v} l_j \left(\sum_{i=1}^{b} n_{ij} B_i \right) \\ &= \frac{1}{r-\lambda} \sum_{j=1}^{v} l_j T_j \\ &= \sum_{j=1}^{v} l_j \tilde{\tau}_j. \end{aligned}$$

The variance of $l'\hat{\tau}$ is obtained as

$$\begin{aligned} Var(l'\hat{\tau}) &= \left(\frac{k}{\lambda v} \right) Var\left(\sum_j l_j Q_j \right) \\ &= \left(\frac{k}{\lambda v} \right)^2 \left[\sum_j l_j^2 Var(Q_j) + 2 \sum_j \sum_{j'(\neq j)} l_j l_{j'} Cov(Q_j, Q_{j'}) \right]. \end{aligned}$$

Since

$$Var(Q_j) = r\left(1 - \frac{1}{k}\right)\sigma^2,$$

$$Cov(Q_j, Q_{j'}) = -\frac{\lambda}{k}\sigma^2, \quad (j \neq j'),$$

so

$$Var(l'\hat{\tau}) = \left(\frac{k}{\lambda v}\right)^2 \left[r\left(1-\frac{1}{k}\right)\sigma^2 \sum_j l_j^2 - \frac{\lambda}{k}\left\{\left(\sum_j l_j\right)^2 - \sum_j l_j^2\right\}\sigma^2 \right]$$

$$= \left(\frac{k}{\lambda v}\right)^2 \left[\frac{r(k-1)}{k}\sum_j l_j^2 + \frac{\lambda}{k}\sum_j l_j^2\right]\sigma^2 \text{ (since } \sum_j \ell_j = 0 \text{ being contrast)}$$

$$= \left(\frac{k}{\lambda v}\right)^2 \frac{1}{k}\left[\lambda(v-1)+\lambda\right]\sum_j l_j^2 \qquad \text{(using } r(k-1)=\lambda(v-1))$$

$$= \left(\frac{k}{\lambda v}\right)\sigma^2 \sum_j l_j^2.$$

Similarly, the variance of $\ell'\hat{\tau}$ is obtained as

$$Var(l'\tilde{\tau}) = \left(\frac{1}{r-\lambda}\right)^2 \left[\sum_j l_j^2 Var(T_j) + 2\sum_j \sum_{j'(\neq j)} l_j l_{j'} Cov(T_j, T_{j'})\right]$$

$$= \left(\frac{1}{r-\lambda}\right)^2 \left[r\sigma_f^2 \sum_j l_j^2 + \lambda\sigma_f^2 \left\{\left(\sum_j l_j\right)^2 - \sum_j l_j^2\right\}\right]$$

$$= \frac{\sigma_f^2}{r-\lambda}\sum_j l_j^2.$$

The information on $\ell'\hat{\tau}$ and $\ell'\hat{\tau}$ can be used together to obtain a more efficient estimator of $\ell'\tau$ by considering the weighted arithmetic mean of $\ell'\hat{\tau}$ and $\ell'\tilde{\tau}$. This will be the minimum variance unbiased and estimator of $\ell'\tau$ when the weights of the corresponding estimates are chosen such that they are inversely proportional to the respective variances of the estimators. Thus the weights to be assigned to intrablock and interblock estimates are reciprocal to their variances as $\lambda v/(k\sigma^2)$ and $(r-\lambda)/\sigma_f^2$, respectively. Then the pooled mean of these two estimators is

$$L^* = \frac{\dfrac{\lambda v}{k\sigma^2} l'\hat{\tau} + \dfrac{r-\lambda}{\sigma_f^2} l'\hat{\tau}}{\dfrac{\lambda v}{k\sigma^2} + \dfrac{r-\lambda}{\sigma_f^2}} = \frac{\dfrac{\lambda v}{k\sigma^2}\sum_j l_j\hat{\tau}_j + \dfrac{r-\lambda}{\sigma_f^2}\sum_j l_j\tilde{\tau}_j}{\dfrac{\lambda v}{k\sigma^2} + \dfrac{r-\lambda}{\sigma_f^2}}$$

$$= \frac{\dfrac{\lambda v\omega_1}{k}\sum_j l_j\hat{\tau}_j + (r-\lambda)\omega_2\sum_j l_j\tilde{\tau}_j}{\dfrac{\lambda v}{k}\omega_1 + (r-\lambda)\omega_2}$$

$$= \frac{\lambda v\omega_1\sum_j l_j\hat{\tau}_j + k(r-\lambda)\omega_2\sum_j l_j\tilde{\tau}_j}{\lambda v\omega_1 + k(r-\lambda)\omega_2}$$

$$= \sum_j l_j\left[\frac{\lambda v\omega_1\hat{\tau}_j + k(r-\lambda)\omega_2\tilde{\tau}_j}{\lambda v\omega_1 + k(r-\lambda)\omega_2}\right]$$

$$= \sum_j l_j\tau_j^*$$

where $\tau_j^* = \dfrac{\lambda v \omega_1 \hat{\tau}_j + k(r-\lambda)\omega_2 \hat{\tau}_j}{\lambda v \omega_1 + k(r-\lambda)\omega_2}$, $\omega_1 = \dfrac{1}{\sigma^2}, \omega_2 = \dfrac{1}{\sigma_f^2}$.

Now we simplify the expression of τ_j^* so that it becomes more compatible in further analysis.

Since $\hat{\tau}_j = (k/\lambda v)Q_j$ and $\tilde{\tau}_j = T_j/(r-\lambda)$, so the numerator of τ_j^* can be expressed as

$$\omega_1 \lambda v \hat{\tau}_j + \omega_2 k(r-\lambda)\tilde{\tau}_j = \omega_1 k Q_j + \omega_2 k T_j$$

Similarly, the denominator of τ_j^* can be expressed as

$$\begin{aligned}
&\omega_1 \lambda v + \omega_2 k(r-\lambda) \\
&= \omega_1 \left[\frac{vr(k-1)}{v-1}\right] + \omega_2 \left[k\left(r - \frac{r(k-1)}{v-1}\right)\right] \qquad (\text{using } \lambda(v-1) = r(k-1)) \\
&= \frac{1}{v-1}\left[\omega_1 vr(k-1) + \omega_2 kr(v-k)\right].
\end{aligned}$$

Let

$$W_j^* = (v-k)V_j - (v-1)T_j + (k-1)G$$

where $\sum_j W_j^* = 0.$ Using these results we have

$$\begin{aligned}
\tau_j^* &= \frac{(v-1)\left[\omega_1 k Q_j + \omega_2 k T_j\right]}{\omega_1 rv(k-1) + \omega_2 kr(v-k)} \\
&= \frac{(v-1)\left[\omega_1(kV_j - T_j) + \omega_2 k T_j\right]}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]} \quad (\text{using } Q_j = V_j - \frac{T_j}{k}) \\
&= \frac{\omega_1 k(v-1)V_j + (k\omega_2 - \omega_1)(v-1)T_j}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]} \\
&= \frac{\omega_1 k(v-1)V_j + (\omega_1 - k\omega_2)\left[W_j^* - (v-k)V_j - (k-1)G\right]}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]} \\
&= \frac{\left[\omega_1 k(v-1) - (\omega_1 - k\omega_2)(v-k)\right]V_j + (\omega_1 - k\omega_2)\left[W_j^* - (k-1)G\right]}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]} \\
&= \frac{1}{r}\left[V_j + \frac{\omega_1 - k\omega_2}{\omega_1 v(k-1) + \omega_2 k(v-k)}\left\{W_j^* - (k-1)G\right\}\right] \\
&= \frac{1}{r}\left[V_j + \xi\left\{W_j^* - (k-1)G\right\}\right]
\end{aligned}$$

where

$$\xi = \frac{\omega_1 - k\omega_2}{\omega_1 v(k-1) + \omega_2 k(v-k)}, \omega_1 = \frac{1}{\sigma^2}, \omega_2 = \frac{1}{\sigma_f^2}.$$

Thus the pooled estimate of the contrast $l'\tau$ is

$$l'\tau^* = \sum_j l_j \tau_j^*$$
$$= \frac{1}{r}\sum_j l_j (V_j + \xi W_j^*) \quad \text{(since } \sum_j l_j = 0 \text{ being contrast)}$$

The variance of $l'\tau^*$ is

$$Var(l'\tau^*) = \frac{k}{\lambda v \omega_1 + k(r-\lambda)\omega_2}\sum_j l_j^2$$
$$= \frac{k(v-1)}{r\left[v(k-1\omega_1 + k(v-k)\omega_2\right]}\sum_j l_j^2 \qquad \text{(using } \lambda(v-1) = r(k-1)\text{)}$$
$$= \sigma_E^2 \frac{\sum_j l_j^2}{r}$$

where

$$\sigma_E^2 = \frac{k(v-1)}{v(k-1)\omega_1 + k(v-k)\omega_2}$$

is called as the **effective variance**.

Note that the variance of any elementary contrast based on the pooled estimates of the treatment effects is

$$Var(\tau_i^* - \tau_j^*) = \frac{2}{r}\sigma_E^2.$$

The effective variance can be approximately estimated by

$$\hat{\sigma}_E^2 = MSE\left[1 + (v-k)\omega^*\right]$$

where MSE is the mean square due to error obtained from the intrablock analysis as

$$MSE = \frac{SS_{Error(t)}}{bk - b - v + 1}$$

and

$$\omega^* = \frac{\omega_1 - \omega_2}{v(k-1)\omega_1 + k(v-k)\omega_2}.$$

The quantity ω^* depends upon the unknown σ^2 and σ_β^2. To obtain an estimate of ω^*, we can obtain the unbiased estimates of σ^2 and σ_β^2 and then substitute them back in place of σ^2 and σ_β^2 in ω^*. To do this, we proceed as follows.

An estimate of ω_1 can be obtained by estimating σ^2 from the intrablock analysis of variance as

$$\hat{\omega}_1 = \frac{1}{\hat{\sigma}_2} = [MSE]^{-1}.$$

The estimate of ω_2 depends on $\hat{\sigma}^2$ and $\hat{\sigma}^2_\beta$. To obtain an unbiased estimator of σ^2_β, consider

$$SS_{Block(adj)} = SS_{Treat(adj)} + SS_{Block(unadj)} - SS_{Treat(unadj)}$$

for which

$$E(SS_{Block(adj)}) = (bk - v)\sigma^2_\beta + (b-1)\sigma^2.$$

Thus an unbiased estimator of σ^2_β is

$$\begin{aligned}
\hat{\sigma}^2_\beta &= \frac{1}{bk-v}\left[SS_{Block(adj)} - (b-1)\hat{\sigma}^2\right] \\
&= \frac{1}{bk-v}\left[SS_{Block(adj)} - (b-1)MSE\right] \\
&= \frac{b-1}{bk-v}\left[MS_{Block(adj)} - MSE\right] \\
&= \frac{b-1}{v(r-1)}\left[MS_{Block(adj)} - MSE\right]
\end{aligned}$$

where

$$MS_{Block(adj)} = \frac{SS_{Block(adj)}}{b-1}.$$

Thus

$$\begin{aligned}
\hat{\omega}_2 &= \frac{1}{k\hat{\sigma}^2 + \hat{\sigma}^2_\beta} \\
&= \frac{1}{v(r-1)\left[k(b-1)SS_{Block(adj)} - (v-k)SS_{Error(t)}\right]}.
\end{aligned}$$

Recall that our main objective is to develop a test of hypothesis for $H_0 : \tau_1 = \tau_2 = ... = \tau_v$ and we now want to develop it using the information based on both interblock and intrablock analysis.

To test the hypothesis related to treatment effects based on the pooled estimate, we proceed as follows.

Consider the adjusted treatment totals based on the intrablock and the interblock estimates as

$$T_j^* = T_j + \omega^* W_j^*; \; j = 1, 2, ..., v$$

and use it as usual treatment total as in earlier cases.

The sum of squares due to T_j^* is

$$S_{T^*}^2 = \sum_{j=1}^{v} T_j^{*2} - \frac{\left(\sum_{j=1}^{v} T_j^*\right)^2}{v}.$$

Note that in the usual analysis of variance technique, the test statistic for such hull hypothesis is developed by taking the ratio of the sum of squares due to treatment divided by its degrees of freedom and the sum of squares due to error divided by its degrees of freedom. Following the same idea, we define the statistics

$$F^* = \frac{S_{T^*}^2 / [(v-1)r]}{MSE[1+(v-k)\hat{\omega}^*]}$$

where $\hat{\omega}^*$ is an estimator of ω^*. It may be noted that F^* depends on $\hat{\omega}^*$. The value of $\hat{\omega}^*$ itself depends on the estimated variances $\hat{\sigma}^2$ and $\hat{\sigma}_f^2$. So it cannot be ascertained that the statistic F^* necessary follow the F distribution. Since the construction of F^* is based on the earlier approaches where the statistic was found to follow the exact F-distribution, so based on this idea, the distribution of F^* can be considered to be approximately F distributed. Thus the approximate distribution of F^* is considered as F distribution with $(v-1)$ and $(bk-b-v+1)$ degrees of freedom. Also, $\hat{\omega}^*$ is an estimator of ω^* which is obtained by substituting the unbiased estimators of ω_1 and ω_2.

An approximate best pooled estimator of $\sum_{j=1} l_j \tau_j$ is

$$\sum_{j=1}^{v} l_j \frac{V_j + \hat{\xi} W_j}{r}$$

and its variance is approximately estimated by

$$\frac{k \sum_j l_j^2}{\lambda v \hat{\omega}_1 + (r-\lambda) k \hat{\omega}_2}.$$

In case of the resolvable BIBD, $\hat{\sigma}_\beta^2$ can be obtained by using the adjusted block with replications sum of squares from the intrablock analysis of variance. If sum of squares due to such block total is SS_{Block}^* and corresponding mean square is

$$MS_{Block}^* = \frac{SS_{Block}^*}{b-r}$$

then

$$E(MS^*_{Block}) = \sigma^2 + \frac{(v-k)(r-1)}{b-r}\sigma^2_\beta$$
$$= \sigma^2 + \frac{(r-1)k}{r}\sigma^2_\beta$$

and $k(b-r) = r(v-k)$ for a resolvable design. Thus

$$E\left[rMS^*_{Block} - MSE\right] = (r-1)(\sigma^2 + k\sigma^2_\beta)$$

and hence

$$\hat{\omega}_2 = \left[\frac{rMS^*_{block} - MSE}{r-1}\right]^{-1},$$
$$\hat{\omega}_1 = \left[MSE\right]^{-1}.$$

The analysis of variance table for the recovery of interblock information in BIBD is described in the following table:

Source	Sum of squares	Degrees of freedom	Mean square	F^*
Between treatment (unadjusted)	$S^2_{T^*}$	v - 1		$F^* = \frac{MS_{Blocks(adj)}}{MSE}$
Between blocks (adjusted)	$SS_{Block(adj)} =$ $SS_{Treat(adj)} +$ $SS_{Block(unadj)-}$ $SS_{Treat(unadj)}$	b - 1	$MS_{Blocks(adj)} = \frac{SS_{Block(adj)}}{b-1}$	
Intrablock error	$SS_{Error(t)}$ (by substraction)	$bk - b - v + 1$	$MSE = \frac{SS_{Error(t)}}{bk-b-v+1}$	
Total	SS_{Total}	bk - 1		

The increase in the precision using interblock analysis as compared to intrablock analysis is

$$\frac{Var(\hat{\tau})}{Var(\tau^*)}-1$$
$$=\frac{\lambda v\omega_1+\omega_2 k(r-\lambda)}{\lambda v\omega_1}-1$$
$$=\frac{\omega_2(r-\lambda)k}{\lambda v\omega_1}.$$

Such an increase may be estimated by

$$\frac{\hat{\omega}_2(r-\lambda)k}{\lambda v\hat{\omega}_1}.$$

Although $\omega_1>\omega_2$ but this may not hold true for $\hat{\omega}_1$ and $\hat{\omega}_2$. The estimates $\hat{\omega}_1$ and $\hat{\omega}_2$ may be negative also and in that case we take $\hat{\omega}_1=\hat{\omega}_2$.