Chapter 9
Confounding

If the number of factors or levels increase in a factorial experiment, then the number of treatment combinations increases rapidly. When the number of treatment combinations is large, then it may be difficult to get the blocks of sufficiently large size to accommodate all the treatment combinations. Under such situations, one may use either connected incomplete block designs, e.g., balanced incomplete block designs (BIBD) where all the main effects and interaction contrasts can be estimated or use unconnected designs where not all these contrasts can be estimated.

Non-estimable contrasts are said to be confounded.

Note that a linear function $\lambda'\beta$ is said to be estimable if there exist a linear function $l'y$ of the observations on random variable y such that $E(l'y) = \lambda'\beta.$ Now there arise two questions. Firstly, what does confounding means and secondly, how does it compares to using BIBD.

In order to understand the confounding, let us consider a simple example of 2^2 factorial with factors a and b. The four treatment combinations are (1), a, b and ab. Suppose each batch of raw material to be used in the experiment is enough only for two treatment combinations to be tested. So two batches of raw material are required. Thus two out of four treatment combinations must be assigned to each block. Suppose this 2^2 factorial experiment is being conducted in a randomized block design. Then the corresponding model is

$$E(y_{ij}) = \mu + \beta_i + \tau_j,$$

then

$$A = \frac{1}{2r}\left[ab + a - b - (1)\right],$$

$$B = \frac{1}{2r}\left[ab + b - a - (1)\right],$$

$$AB = \frac{1}{2r}\left[ab + (1) - a - b\right].$$

Suppose the following block arrangement is opted:

Block 1	Block 2
(1) ab	a b

The block effects of blocks 1 and 2 are β_1 and β_2, respectively, then the average responses corresponding to treatment combinations a, b, ab and (1) are

$$\begin{aligned}
&E\left[y(a)\right]=\mu+\beta_2+\tau(a),\\
&E\left[y(b)\right]=\mu+\beta_2+\tau(b),\\
&E\left[y(ab)\right]=\mu+\beta_1+\tau(ab),\\
&E\left[y(1)\right]=\mu+\beta_1+\tau(1),
\end{aligned}$$

respectively. Here $y(a),\ y(b),\ y(ab),\ y(1)$ and $\tau(a),\ \tau(b),\ \tau(ab),\ \tau(1)$ denote the responses and treatments corresponding to a, b, ab and (1), respectively. Ignoring the factor $1/2r$ in A, B, AB and using $E[y(a)],\ E[y(b)],\ E[y(ab)],\ E(y(1)]$, the effect A is expressible as follows :

$$\begin{aligned}
A&=[\mu+\beta_1+\tau(ab)]+[\mu+\beta_2+\tau(a)]-[\mu+\beta_2+\tau(b)]-[\mu+\beta_1+\tau(1)]\\
&=\tau(ab)+\tau(a)-\tau(b)-\tau(1).
\end{aligned}$$

So the block effect is not present in A and it is not mixed up with the treatment effects. In this case, we say that the main effect A is not confounded. Similarly, for the main effect B, we have

$$\begin{aligned}
B&=[\mu+\beta_1+\tau(ab)]+[\mu+\beta_2+\tau(b)]-[\mu+\beta_2+\tau(a)]-[\mu+\beta_1+\tau(1)]\\
&=\tau(ab)+\tau(b)-\tau(a)-\tau(1).
\end{aligned}$$

So there is no block effect present in B and thus B is not confounded. For the interaction effect AB, we have

$$\begin{aligned}
AB&=[\mu+\beta_1+\tau(ab)]+[\mu+\beta_1+\tau(1)]-[\mu+\beta_2+\tau(a)]-[\mu+\beta_2+\tau(b)]\\
&=2(\beta_1-\beta_2)+\tau(ab)+\tau(1)-\tau(a)-\tau(b).
\end{aligned}$$

Here the block effects are present in AB. In fact, the block effects are β_1 and β_2 are mixed up with the treatment effects and cannot be separated individually from the treatment effects in AB. So AB is said to be confounded (or mixed up) with the blocks.

Alternatively, if the arrangement of treatments in blocks are as follows:

Block 1	Block 2
ab a	(1) b

then the main effect A is expressible as

$$\begin{aligned} A &= [\mu+\beta_1+\tau(ab)]+[\mu+\beta_1+\tau(a)]-[\mu+\beta_2+\tau(b)]-[\mu+\beta_2+\tau(1)] \\ &= 2(\beta_1-\beta_2)+\tau(ab)+\tau(a)-\tau(b)-\tau(1) \end{aligned}$$

Observe that the block effects β_1 and β_2 are present in this expression. So the main effect A is confounded with the blocks in this arrangement of treatments.

So the main effect A is confounded with the blocks in this arrangement of treatments.

We notice that it is in our control to decide that which of the effect is to be confounded. The order in which treatments are run in a block is determined randomly. The choice of block to be run first is also randomly decided.

The following observation emerges from the allocation of treatments in blocks:

"*For a given effect, when two treatment combinations with the same signs are assigned to one block and the other two treatment combinations with the same but opposite signs are assigned to another block, then the effect gets confounded*".

For example, in case AB is confounded, then

- ab and (1) with + signs are assigned to block 1 whereas
- a and b with – signs are assigned to block 2.

Similarly, when A is confounded, then

- a and ab with + signs are assigned to block 1 whereas
- (1) and b with – signs are assigned to block 2.

The reason behind this observation is that if every block has treatment combinations in the form of linear contrast, then effects are estimable and thus unconfounded. This is also evident from the theory of linear estimation that a linear parametric function is estimable if it is in the form of a linear contrast.

The contrasts which are not estimable are said to be **confounded with the differences between blocks** (or block effects). The contrasts which are estimable are said to be unconfounded with blocks or free from block effects.

Comparison of balanced Incomplete Block design (BIBD versus factorial:

Now we explain how confounding and BIBD compares together. Consider a 2^3 factorial experiment which needs the block size to be 8. Suppose the raw material available to conduct the experiment is sufficient only for a block of size 4. One can use BIBD in this case with parameters $b = 14, k = 4, v = 8, r = 7$ and $\lambda = 3$ (such BIBD exists). For this BIBD, the efficiency factor is

$$E = \frac{\lambda v}{kr} = \frac{6}{8}$$

and

$$Var(\hat{\tau}_j - \hat{\tau}_{j'})_{BIBD} = \frac{2k}{\lambda v}\sigma^2 = \frac{2}{6}\sigma^2 \quad (j \neq j').$$

Consider now an unconnected design in which 7 out of 14 blocks get treatment combination in block 1 as

$$\boxed{a \;\; b \;\; c \;\; abc}$$

and remaining 7 blocks get treatment combination in block 2 as

$$\boxed{(1) \;\; ab \;\; bc \;\; ac}$$

In this case, all the effects A, B, C, AB, BC and AC are estimable but ABC is not estimable because the treatment combinations with all + and all – signs in

$$\begin{aligned} ABC &= (a-1)(b-1)(c-1) \\ &= \underbrace{(a+b+c+abc)}_{\text{in block 1}} - \underbrace{((1)+ab+bc+ac)}_{\text{in block 2}} \end{aligned}$$

are contained in same blocks. In this case, the variance of estimates of unconfounded main effects and interactions is $8\sigma^2/7$. Note that in case of RBD,

$$Var(\hat{\tau}_j - \hat{\tau}_{j'})_{RBD} = \frac{2\sigma^2}{r} = \frac{2\sigma^2}{7} \quad (j \neq j')$$

and there are four linear contrasts, so the total variance is $4 \times (2\sigma^2/7)$ which gives the factor $8\sigma^2/7$ and which is smaller than the variance under BIBD.

We observe that at the cost of not being able to estimate ABC, we have better estimates of A, B, C, AB, BC and AC with the same number of replicates as in BIBD. Since higher order interactions are difficult to interpret and are usually not large, so it is much better to use confounding arrangements which provide better estimates of the interactions in which we are more interested.

Note that this example is for understanding only. As such the concepts behind incomplete block design and confounding are different.

Confounding arrangement:

The arrangement of treatment combinations in different blocks, whereby some pre-determined effect (either main or interaction) contrasts are confounded is called a confounding arrangement.

For example, when the interaction ABC is confounded in a 2^3 factorial experiment, then the confounding arrangement consists of dividing the eight treatment combinations into following two sets:

$$\boxed{a\ \ b\ \ c\ \ abc}$$

and

$$\boxed{(1)\ \ ab\ \ bc\ \ ac}$$

With the treatments of each set being assigned to the same block and each of these sets being replicated same number of times in the experiment, we say that we have a confounding arrangement of a 2^3 factorial in two blocks. It may be noted that any confounding arrangement has to be such that only predetermined interactions are confounded and the estimates of interactions which are not confounded are orthogonal whenever the interactions are orthogonal.

Defining contrast:

The interactions which are confounded are called the defining contrasts of the confounding arrangement.

A confounded contrast will have treatment combinations with the same signs in each block of the confounding arrangement. For example, if effect $AB = (a-1)(b-1)(c+1)$ is to be confounded, then put all factor combinations with + sign, i.e., (1), ab, c and abc in one block and all other factor combinations with – sign, i.e., a, b, ac and bc in another block. So the **block size reduces to 4 from 8** when one effect is confounded in 2^3 factorial experiment.

Suppose if along with ABC confounded, we want to confound C also,. To obtain such blocks, consider the blocks where ABC is confounded and divide them into further halves. So the block

$$\boxed{a\ \ b\ \ c\ \ abc}$$

is divided into following two blocks: $\boxed{a\ \ b}$ and $\boxed{c\ \ abc}$

and the block

$\boxed{(1)\ \ ab\ \ bc\ \ ac}$

is divided into following two blocks: $\boxed{(1)\ \ ab}$ and $\boxed{bc\ \ ac}$

These blocks of 4 treatments are divided into 2 blocks with each having 2 treatments and they are obtained in the following way. If only C is confounded then the block with + sign of treatment combinations in C is

$\boxed{c\ \ ac\ \ bc\ \ abc}$

and block with – sign of treatment combinations in C is

$\boxed{(1)\ \ a\ \ b\ \ ab}$.

Now look into the

(i) following block with + sign when $ABC = (a-1)(b-1)(c-1)$ is confounded,

$\boxed{a\ \ b\ \ c\ \ abc}$

(ii) following block with + sign when $C = (a+1)(b+1)(c-1)$ is confounded and

$\boxed{c\ \ ab\ \ ac\ \ abc}$

(iii) table of + and – signs in case of 2^3 factorial experiment.

Identify the treatment combinations having common - signs in these two blocks in (i) and (ii). These treatment combinations are are c and abc. So assign them into one block. The remaining treatment combinations out of a, b, c and abc are a and b which go into another block.

Similarly look into the

(a) following block with – sign when ABC is confounded,

$\boxed{(1)\ \ ab\ \ bc\ \ ac}$

(b) following block with – sign when C is confounded and

$\boxed{(1)\ \ a\ \ b\ \ ab}$

(c) table of + and – signs in case of 2^3 factorial experiment.

Identify the treatment combinations having common – sign in these two blocks in (a) and (b). These treatment combinations are (1) and ab which go into one block and the remaining two treatment combinations ac and bc out of c, ac, bc and abc go into another block. So the blocks where both ABC and C are confounded together are

$\boxed{(1)\ \ ab}$, $\boxed{a\ \ b}$, $\boxed{ac\ \ bc}$ and $\boxed{c\ \ abc}$.

While making these assignments of treatment combinations into four blocks, each of size two, we notice that another effect, viz., AB also gets confounded automatically. Thus we see that when we confound two factors, a third factor is automatically getting confounded. This situation is quite general. The defining contrasts for a confounding arrangement cannot be chosen arbitrarily. If some defining contrasts are selected then some other will also get confounded.

Now we present some definitions which are useful in describing the confounding arrangements.

Generalized interaction:

Given any two interactions, the generalized interaction is obtained by multiplying the factors (in capital letters) and ignoring all the terms with an even exponent.

For example, the generalized interaction of the factors ABC and BCD is $ABC \times BCD = AB^2C^2D = AD$ and the generalized interaction of the factors AB, BC and ABC is $AB \times BC \times ABC = A^2B^3C^2 = B.$

Independent set :

A set of main effects and interaction contrasts is called independent if no member of the set can be obtained as a generalized interaction of the other members of the set.

For example, the set of factors AB, BC and AD is an independent set but the set of factors AB, BC, CD and AD is not an independent set because $AB \times BC \times CD = AB^2C^2D = AD$ which is already contained in the set.

Orthogonal treatment combinations:

The treatment combination $a^p b^q c^r \ldots$ is said to be orthogonal to the interaction $A^x B^y C^z ....$ if $(px + qy + rz + ...)$ is divisible by 2. Since $p, q, r, ..., x, y, z, ...$ are either 0 or 1, so a treatment combination is orthogonal to an interaction if they have an even number of letters in common.Treatment combination (1) is orthogonal to every interaction.

If $a^{p_1} b^{q_1} c^{r_1}$ ....and $a^{p_2} b^{q_2} c^{r_2} ...$ are both orthogonal to $A^x B^y C^z ...,$ then the product $a^{p_1+p_2} b^{q_1+q_2} c^{r_1+r_2} ...$ is also orthogonal to $A^x B^y C^z ...$ Similarly, if two interactions are orthogonal to a treatment combination, then their generalized interaction is also orthogonal to it.

Now we give some general results for a confounding arrangement. Suppose we wish to have a confounding arrangement in 2^p blocks of a 2^k factorial experiment. Then we have the following observations:

1. The size of each block is 2^{k-p}.
2. The number of elements in defining contrasts is $(2^p - 1), i.e., (2^p - 1)$ interactions have to be confounded.

   Proof: If p factors are to be confounded, then the number of mth order interaction with p factors is $\binom{p}{m}$, $(m = 1, 2, ..., p)$. So the total number of factors to be confounded are $\sum_{m=1}^{p} \binom{p}{m} = 2^{p-1}$.
3. If any two interactions are confounded, then their generalized interactions are also confounded.
4. The number of independent contrasts out of $(2^p - 1)$ defining contrasts is p and rest are obtained as generalized interactions.
5. Number of effects getting confounded automatically is $(2^p - p - 1)$.

To illustrate this, consider a 2^5 factorial $(k = 5)$ with 5 factors, viz., A, B, C, D and E. The factors are to be confounded in 2^3 blocks ($p = 3$). So the size of each block is $2^{5-3} = 4$. The number of defining contrasts is $2^3 - 1 = 7$. The number of independent contrasts which can be chosen arbitrarily is 3(i.e., p) out of 7 defining contrasts. Suppose we choose $p = 3$ following independent contrasts as

(i) ACE

(ii) CDE

(iii) $ABDE$

and then the remaining 4 out of 7 defining contrasts are obtained as

(iv) $(ACE) \times (CDE) = AC^2DE^2 = AD$

(v) $(ACE) \times (ABDE) = A^2BCDE^2 = BCD$

(vi) $(CDE) \times (ABDE) = ABCD^2E^2 = ABC$

(vii) $(ACE) \times (CDE) \times (ABDE) = A^2BC^2D^2E^3 = BE$.

Alternatively, if we choose another set of $p = 3$ independent contrast as

(i) $ABCD$,

(ii) $ACDE$,

(iii) $ABCDE$,

then the defining contrasts are obtained as

(iv) $(ABCD)\times(ACDE) = A^2BC^2D^2E = BE$

(v) $(ABCD)\times(ABCDE) = A^2B^2C^2D^2E = E$

(vi) $(ACDE)\times(ABCDE) = A^2BC^2D^2E^2 = B$

(vii) $(ABCD)\times(ACDE)\times(ABCDE) = A^3B^2C^3D^3E^2 = ACD.$

In this case, the main effects B and E also get confounded.

As a rule, try to confound, as far as possible, higher order interactions only because they are difficult to interpret.

After selecting p independent defining contrasts, divide the 2^k treatment combinations into 2^p groups of 2^{k-p} combinations each, and each group going into one block.

Principal (key) block:

Group containing the combination (1) is called the principal block or key block. It contains all the treatment combinations which are orthogonal to the chosen independent defining contrasts.

If there are p independent defining contrasts, then any treatment combination in principal block is orthogonal to p independent defining contrasts. In order to obtain the principal block,

- write the treatment combinations in standard order.
- check each one of them for orthogonality.
- If two treatment combinations belongs to the principal block, their product also belongs to the principal block.
- When few treatment combinations of the principal block have been determined, other treatment combinations can be obtained by multiplication rule.

Now we illustrate these steps in the following example.

Example:

Consider the set up of a 2^5 factorial experiment in which we want to divide the total treatment effects into 2^3 groups by confounding three effects AD, BE and ABC. The generalized interactions in this case are $ADBE$, BCD, ACE and CDE.

In order to find the principal block, first write the treatment combinations in standard order as follows:

(1)	a	b	ab	c	ac	bc	abc
d	ad	bd	abd	cd	acd	bcd	$abcd$
e	ae	be	abe	ce	ace	bce	$abce$
de	ade	bde	$abde$	cde	$acde$	$bcde$	$abcde$.

Place a treatment combination in the principal block if it has an even number of letters in common with the confounded effects AD, BE and ABC. The principal block has $(1), acd$, bce and $abde (= acd \times bce)$. Obtain other blocks of confounding arrangement from principal block by multiplying the treatment combinations of the principal block by a treatment combination not occurring in it or in any other block already obtained. In other words, choose treatment combinations not occurring in it and multiply with them in the principal block. Choose only distinct blocks. In this case, obtain other blocks by multiplying a, b, ab, c, ac, bc, abc like as in the following .

Arrangement of the treatments in blocks when AD, BE and ABC are confounded

Principal Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8
(1)	a	b	ab	c	ac	bc	abc
acd bce $abde$	cd $abce$ bde	$abcd$ ce ade	bcd ace de	ad be $abcde$	d abe $bcde$	abd e $acde$	bd ae cde

For example, block 2 is obtained by multiplying a with each factor combination in principal block as $(1) \times a = a$, $acd \times a = a^2cd = cd, bce \times a = abce$, $abde \times a = a^2bde = bde$; block 3 is obtained by multiplying b with (1), acd, bce and $abde$ and similarly other blocks are obtained. If any other treatment combination is chosen to be multiplied with the treatments in principal block, then we get a block which will be one among the blocks 1 to 8. For example, if ae is multiplied with the treatments in principal block, then the blocks obtained consists of $(1)\times ae = ae,\ acd\times ae = cde,\ bce\times ae = abc$ and $abde\times ae = bd$ which is same as the block 8.

Alternatively, if $ACD, ABCD$ and $ABCDE$ are to be confounded, then independent defining contrasts are ACD, $ABCD$, $ABCDE$ and the principal block has (1), ac, ad and $cd(= ac\times ad)$.

Analysis of variance in case of confounded effects

When an effect is confounded, it means that it is not estimable. The following steps are followed to conduct the analysis of variance in case of factorial experiments with confounded effects:

- Obtain the sum of squares due to main and interaction effects in the usual way as if no effect is confounded.
- Drop the sum of squares corresponding to confounded effects and retain only the sum of squares due to unconfounded effects.
- Find the total sum of squares.
- Obtain the sum of squares due to error and associated degrees of freedom by subtraction.
- Conduct the test of hypothesis in the usual way.