CONSISTENT ESTIMATION THROUGH WEIGHTED HARMONIC MEAN OF INCONSISTENT ESTIMATORS IN REPLICATED MEASUREMENT ERROR MODELS

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ABSTRACT

The present paper proposes a consistent estimator of slope parameter in a replicated measurement error model arising from the weighted harmonic average of two estimators which are themselves inconsistent.

1. INTRODUCTION

Traditional measurement error models do not lend themselves to consistent estimation of parameters unless some additional information besides sample observation is available. This additional information may comprise, for instance, availability of instrumental variable or knowledge of some parameter such as intercept term, one or both the error variances or their ratios; see, e.g., (1), (2) and (3) for a comprehensive exposition. Even when replicated observations are available, the two estimators of slope parameter arising from application of least squares procedure employing original observations and aggregated (over replications) observations are found to be inconsistent, see, (4).

In order to obtain a consistent estimator from a combination of two inconsistent estimators, we consider and show that a weighted harmonic average provides a consistent estimator. An elegant aspect of this estimator is that the weights assigned to two inconsistent estimators do not depend upon any unknown parameter and the resulting estimator has a simple form.
2. CONSISTENT ESTIMATION

Following (5), we postulate the replicated ultrastructural version of a measurement error model:

\[ Y_i = \alpha + \beta X_i \]  \hspace{1cm} (2.1)

\[ y_{ij} = Y_i + u_{ij} \]  \hspace{1cm} (2.2)

\[ x_{ij} = X_i + v_{ij} \]  \hspace{1cm} (2.3)

with \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, r \).

Here \( Y_i \) and \( X_i \) denote the true values that are unobserved. Instead, \( r \) replicated observations on them are available but they are contaminated by measurement errors \( u_{ij} \) and \( v_{ij} \) respectively. The unknown parameters are \( \alpha \) and \( \beta \).

Further, we write

\[ X_i = m_i + w_i \quad (i = 1, 2, \ldots, n) \]  \hspace{1cm} (2.4)

where \( m_1, m_2, \ldots, m_n \) are fixed while \( w_1, w_2, \ldots, w_n \) are random variables assumed to be independently and identically distributed with mean 0 and variance \( \sigma_w^2 \). Thus when \( m_1, m_2, \ldots, m_n \) are equal, we have the specification of a structural model. When \( \sigma_w^2 = 0 \), we have the functional model.

It is assumed that \( u_{ij} \)’s are independently and identically distributed with mean 0 and finite variance. Similarly, all \( v_{ij} \)’s are independently and identically distributed with mean 0 and variance \( \sigma_v^2 \). Further, all \( w_i \)’s, \( u_{ij} \)’s and \( v_{ij} \)’s are assumed to be mutually independent.

If we use the observations \( x_{ij} \)’s and \( y_{ij} \)’s, the least squares estimator of \( \beta \) is given by

\[ b = \frac{\sum_i \sum_j (x_{ij} - \bar{x})(y_{ij} - \bar{y})}{\sum_i \sum_j (x_{ij} - \bar{x})^2} \]  \hspace{1cm} (2.5)

where \( \bar{x} = (1/nr) \sum_i \sum_j x_{ij} \) and \( \bar{y} = (1/nr) \sum_i \sum_j y_{ij} \).

On the other hand, if we employ the averages \( \bar{x}_i \) and \( \bar{y}_i \) taken over replications, the least squares estimator of \( \beta \) is

\[ b^* = \frac{\sum_i (\bar{x}_i - \bar{x})(\bar{y}_i - \bar{y})}{\sum_i (\bar{x}_i - \bar{x})^2}. \]  \hspace{1cm} (2.6)

In order to study the asymptotic properties, we assume that the limiting value of variance of \( m_1, m_2, \ldots, m_n \) as \( n \) tends to infinity is \( \sigma_m^2 \) which is finite. Further, we assume that \( n \) grows large while \( r \) stays fixed.
Using the model specification, it can be easily seen that

\[ \lim_{n \to \infty} b = \frac{\sigma_m^2 + \sigma_w^2}{\sigma_v^2 + \sigma_m^2 + \sigma_w^2} \beta \]  

(2.7)

\[ \lim_{n \to \infty} b^* = \frac{(\sigma_m^2 + \sigma_w^2)r}{\sigma_v^2 + (\sigma_m^2 + \sigma_w^2)r} \beta \]  

(2.8)

from which it is obvious that both the estimators \( b \) and \( b^* \) are inconsistent.

For the development of consistent estimators of \( \beta \), a popular device is to combine the two inconsistent estimators in such a manner that the resulting estimator is consistent. For instance, if we consider a linear combination \([c_0 b^* + (1 - c_0) b]\) and choose \(0 \leq c_0 \leq 1\) so that it is a consistent estimator of \( \beta \), we find

\[ c_0 = -\frac{\sigma_v^2 + \sigma_m^2 + \sigma_w^2}{(r-1)(\sigma_m^2 + \sigma_w^2)} : \quad r \geq 2. \]

Clearly, such a choice of \( c_0 \) is not interesting owing to involvement of unknown variances. However, if we take a non-linear combination such as weighted harmonic average of \( b \) and \( b^* \), we have

\[ \frac{1}{b_H} = \frac{c}{b^*} + \frac{1-c}{b} \]

where \(0 \leq c \leq 1\) is the weight assigned to \( b^* \).

This non-linear combination serves as a consistent estimator of \( \beta \) in the sense that

\[ \lim_{n \to \infty} (b_H - \beta) = 0 \]

when

\[ c = \frac{r}{r-1} : \quad r \geq 2. \]  

(2.9)

This yields the following weighted harmonic mean estimator of \( \beta \):

\[ \frac{1}{b_H} = \left( \frac{1}{r-1} \right) \left( \frac{r}{b^*} - \frac{1}{b} \right) \]

or

\[ b_H = \frac{(r-1)bb^*}{rb - b^*}. \]  

(2.10)

It can be readily verified that \( b_H \) is a consistent estimator of \( \beta \). An interesting feature of such a combined estimator when compared with the linearly combined estimators is that the weights do not involve any unknown quantity and they have a simple form.
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REFERENCES
