PROPERTIES OF A CONSISTENT ESTIMATION PROCEDURE IN ULTRASTRUCTURAL MODEL WHEN RELIABILITY RATIO IS KNOWN

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(Received for publication 1 August 1995)

Abstract—To overcome the problem of inconsistency of the ordinary least squares (OLS) estimators for regression coefficients in a linear measurement error model, we assume the reliability ratio to be known a priori and utilize this information to form consistent estimators in an ultrastructural model. Assuming the distributions of measurement errors and the random error component to be not necessarily normal, the efficiency properties of the estimator for slope parameter have been analysed and the effect of departure from normality is examined. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION
The least squares procedure is well known to yield the minimum variance estimator in the class of linear and unbiased estimators for the coefficients in a linear regression model. This optimal property implicitly assumes that the observations on variables are correctly measured. Such a specification is often not tenable in many scientific investigations and the observations are invariably subject to measurement errors. When these errors are negligibly small they may not appreciably alter the inferences deduced from the application of least squares. On the other hand, when the measurement errors are not negligibly small, the inferences are significantly altered and in fact become invalid. The main reason is that the estimators arising from least squares procedure are not even consistent in the presence of measurement errors.

The quest for consistent estimation has received considerable attention in the literature and has led to the evaluation of several estimators but little is known about their properties, particularly when the errors are not necessarily normally distributed; see e.g. Fuller [1]. The plan of this paper is as follows. In Section 2, we consider an ultrastructural formulation of the linear regression model with measurement errors; see Dolby [2]. In Section 3, we describe a consistent estimation procedure when reliability ratio is known, while in Section 4 we analyse the large sample asymptotic properties of this procedure. Finally, Section 5 draws some conclusions. In the end, some expectations are derived in the Appendix.

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2. MODEL SPECIFICATION
Let us consider the following linear ultrastructural model

\[ Y_i = \alpha + \beta X_i \quad (i = 1, 2, \ldots, n) \quad (1) \]
\[ y_i = Y_i + u_i \quad (2) \]
\[ x_i = X_i + v_i \quad (3) \]
\[ X_i = m_i + w_i \quad (4) \]

where \( \alpha \) is the intercept term and \( \beta \) is the slope parameter. Here, \( y_i \) and \( x_i \) are the observed values corresponding to true values of the study variable \( Y_i \) and explanatory variable \( X_i \), respectively; \( u_i \) and \( v_i \) are the measurement errors associated with the study variable and explanatory variable, respectively; and \( w_i \) is the random error component associated with the explanatory variable.

Let us make some assumptions about the error terms. \( u_1, u_2, \ldots, u_n \) are assumed to be identically and independently distributed with mean 0 and variance \( \sigma_u^2 \), along with third and fourth central moments \( \gamma_1 u, \sigma_u^3 \) and \( (\gamma_2 u + 3)\sigma_u^4 \), respectively. Similarly, \( v_1, v_2, \ldots, v_n \) are also assumed to be identically and independently distributed with mean 0 and variance \( \sigma_v^2 \) along with third and fourth central moments \( \gamma_1 v, \sigma_v^3 \) and \( (\gamma_2 v + 3)\sigma_v^4 \), respectively. The random error components \( w_1, w_2, \ldots, w_n \) are also assumed to be identically and independently distributed with mean 0, variance \( \sigma_w^2 \), third central moment as \( \gamma_1 w, \sigma_w^3 \) and the fourth central moment as \( (\gamma_2 w + 3)\sigma_w^4 \). Here \( \gamma_1 \) and \( \gamma_2 \) are the Pearson measure of excess of skewness and kurtosis, respectively in the error distribution of the variable in the suffix.

It is further assumed that \( u_1, u_2, \ldots, u_n; v_1, v_2, \ldots, v_n \) and \( w_1, w_2, \ldots, w_n \) are mutually independent of each other.
It may be noticed here that no specific form of the error distribution has been assumed. All that is assumed is the existence of first four moments of the error distributions.

To study the efficiency behaviour of estimators in a comprehensive manner, we introduce the following quantities:

\[ d = \frac{\sigma_w^2}{S_{mm} + \sigma_w^2}; \quad 0 \leq d < 1, \quad (5) \]

where

\[ S_{mm} = \frac{1}{n} \sum (m_i - \bar{m})^2 \quad \text{with} \quad \bar{m} = \frac{1}{n} \sum m_i. \quad (6) \]

It is observed that in case of functional form of the measurement error model, the \( X_i \)s are all fixed, so that \( w = 0 \) for all \( i = 1, 2, \ldots, n \) and consequently \( \sigma_w^2 = 0 \). This makes \( d \) equal to 0. Similarly, \( d \) is equal to 1 in case of structural form of the measurement error model because all means of \( X_i \)s are the same implying that \( S_{mm} = 0 \). Thus, \( d \) serves as a measure of departure of ultrastructural model from its two forms, namely the functional and the structural.

In order to study asymptotic properties, the \( m_1, m_2, \ldots, m_n \) are assumed to be asymptotically cooperative in the sense that \( \bar{m} = n^{-1} \sum m_i \) and \( S_{mm} = n^{-1} \sum (m_i - \bar{m})^2 \) tend to finite limits as \( n \) grows large. This specification is required to avoid the presence of any trend; see Schneeweiss [3].

3. A CONSISTENT ESTIMATION PROCEDURE

Combining (1), (2) and (3), we get

\[ y_i = \alpha + \beta x_i + (u_i - \beta v_i). \quad (7) \]

Writing

\[ s_{xx} = \frac{1}{n} \sum (x_i - \bar{x})^2; \quad \bar{x} = \frac{1}{n} \sum x_i \]

\[ s_{xy} = \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y}); \quad \bar{y} = \frac{1}{n} \sum y_i \]

the least squares estimator of \( \beta \) is given by

\[ \hat{\beta} = \frac{s_{xy}}{s_{xx}} \quad (9) \]

which is a consistent estimator and possesses minimum variance in the class of linear and unbiased estimators of \( \beta \) provided that the observations on explanatory variable are free from any measurement error. When these observations are contaminated by measurement errors, it is well known that \( \hat{\beta} \) is neither consistent nor unbiased.

The problem of inconsistency of \( \hat{\beta} \) is circumvented through the use of some additional information; see e.g. Fuller [1] for a comprehensive account. Among them, an interesting situation arises when this information pertains to the reliability ratio defined by

\[ \rho = \frac{S_{nm} + \sigma_x^2}{S_{mm} + \sigma_x^2 + \sigma_w^2}, \quad (10) \]

which is indeed the ratio of variance of \( X \) to variance of its observed counterpart \( x \).

In many scientific investigations, the value of \( \rho \) is correctly known. Such a knowledge may arise from some theoretical considerations or from empirical experience. Or a reasonably accurate estimate of \( \rho \) may be available from some other independent studies. Moreover, the reliability ratio is not difficult to find in many applications and there are various procedures to do it; see e.g. Ashley and Vanhan [4], Lord and Novick [5] and Marquis et al. [6].

Assuming \( \rho \) to be known, a correction for removing the contamination of measurement errors is applied to the least squares estimator; see Fuller ([1], p. 5). This yields the following consistent estimator of \( \beta \):

\[ b = \frac{s_{xy}}{\rho s_{xx}} \quad (11) \]

which is termed as attenuated or adjusted or immaculate estimator.

4. PROPERTIES OF ESTIMATOR

The asymptotic properties of the estimator \( b \) have been studied by Fuller and Hidiroglou [7] for the structural variant of ultrastructural model under the assumption of normality. We, however, assume that errors are not necessarily normally distributed, and consider a general model.

In order to study the large sample asymptotic properties of the consistent estimator \( b \), let

\[ Z_{uv} = n^{-1/2}u'Av - n^{-1/2}a_i^2 \]

\[ Z_{uw} = n^{-1/2}w'Aw - n^{-1/2}a_i^2 \]

\[ Z_{mu} = n^{-1/2}m'Au; \quad Z_{mu} = n^{-1/2}m'Au \]

\[ Z_{uw} = n^{-1/2}u'Aw; \quad Z_{uw} = n^{-1/2}u'Aw \]

where

\[ u = \text{Col.}(u_1, u_2, \ldots, u_n) \]

\[ v = \text{Col.}(v_1, v_2, \ldots, v_n) \]

\[ w = \text{Col.}(w_1, w_2, \ldots, w_n) \]

\[ e = \text{Col.}(1, 1, \ldots, 1) \]

\[ A = I - ee' \]

Further, we introduce the following notation:

\[ r = \frac{\sigma_w^2}{\rho \beta^2 \sigma_x^2}; \quad 0 < r < \infty \quad (13) \]

\[ h_x = (1 - \rho)\sigma_x^{-2}[Z_{uv} + Z_{uw} + 2(Z_{mu} + Z_{mu} + Z_{uw})] \quad (14) \]

\[ h_y = \sigma_x^{-2}[(1 - \rho)Z_{uv} - \rho Z_{uv} + (1 - 2\rho)(Z_{mu} + Z_{uw} + Z_{uw}) + 2(1 - \rho)Z_{uw} + \beta^{-1}(Z_{mu} + Z_{uw} + Z_{uw})]. \quad (15) \]
Employing eqns (1)-(4), we can express (16) as follows:

\[
\frac{b - \beta}{\beta} = n^{-1/2} \left( \frac{1 - \rho}{\rho} \right) \left[ 1 + n^{-1/2}h_2 \right]^{-1}h_y
\]

\[
= \frac{1 - \rho}{\rho} \left[ n^{-1/2}h_y - n^{-1}h_xh_y \right] + O_p(n^{-2}).
\]

(16)

Thus, the relative bias to order \(O(n^{-1})\) is given by

\[
RB(b) = \frac{1 - \rho}{\rho} \left[ n^{-1/2}E(h_y) - n^{-1}E(h_xh_y) \right].
\]

(17)

Similarly, the relative mean squared error to order \(O(n^{-1})\) is

\[
RM(b) = \frac{1 - \rho}{\rho} \left[ \frac{1 - \rho}{\rho} \right]^2 E(h_y^2).
\]

(18)

Substituting the expected values of \(h_y, h_xh_y\) and \(h_y^2\) from the Appendix, we obtain the following results.

**Theorem**

The large sample asymptotic approximations for the relative bias and relative mean squared error of \(b\) up to order \(O(n^{-1})\) are given by

\[
RB(b) = \left( \frac{1 - \rho}{\rho} \right) \left[ B + (1 - \rho)\gamma_{2e} + d^2\gamma_{2w} \right]
\]

(19)

\[
RM(b) = \left( \frac{1 - \rho}{\rho} \right)^2 \left[ M + \gamma_{2e} + d^2\gamma_{2w} \right].
\]

(20)

where

\[
B = (1 - d) + 2\rho(1 - d)^2
\]

(21)

\[
M = \frac{1 + r}{\rho(1 - \rho)} - 2(1 - d)^2.
\]

(22)

An interesting observation emerging from the expressions (19) and (20) is that only the relative peakedness of the error distributions associated with the explanatory variable influences the efficiency properties of the estimator; the skewness has no role to play. Further, the skewness and kurtosis of the error distribution associated with study variable exhibits no influence at all on the properties of estimator, at least to the order of our approximation.

When the distributions of \(v\) and \(w\) are mesokurtic (or normal) so that \(\gamma_{2e} = \gamma_{2w} = 0\), the expressions (19) and (20) reduce to the following:

\[
RB(b) = \frac{1 - \rho}{n} B
\]

(23)

\[
RM(b) = \frac{(1 - \rho)^2}{n} M.
\]

(24)

Equation (23) is also obtained when the distributions are not mesokurtic but satisfy the constraint:

\[
\frac{\gamma_{2e}}{\gamma_{2w}} = \frac{\rho^2}{1 - \rho}.
\]

(25)

Further, from (23), we notice that the relative bias is always positive. This bias increases when

\[
\frac{\gamma_{2e}}{\gamma_{2w}} > \frac{\rho^2}{1 - \rho}
\]

(26)

while it may decrease when the inequality (26) holds true with a reversed sign.

Similarly, we observe from (20) and (24) that the relative mean squared error to the order of our approximation may decline making \(b\) more efficient than in the mesokurtic or normal case when both the distributions of \(v\) and \(w\) are platykurtic. If both the distributions are leptokurtic, the efficiency declines. When one is platykurtic and the other is leptokurtic, the net effect on the efficiency depends upon the value of the factor \((\gamma_{2e} + d^2\gamma_{2w})\). If this factor is negative, the precision increases while it decreases when this factor is positive.

The above discussion brings out the effect of departure from normality of the error distributions on the efficiency properties of the consistent estimator \(b\).

Next, let us consider the two popular variants of the ultrastructural model, namely the structural and functional which are the two extremes of the ultrastructural model. Setting \(d = 0\) and \(d = 1\), we find

\[
RB(b) = \frac{1 - \rho}{n} \left[ 1 + 2\rho + \gamma_{2e} \right]
\]

(27)

\[
RB(b) = \frac{1 - \rho}{n} \left[ (1 - \rho)\gamma_{2e} - \rho\gamma_{2w} \right]
\]

(28)

\[
RB(b) = \frac{(1 - \rho)^2}{n} \left[ 1 + 2\rho + \gamma_{2e} \right]
\]

(29)

\[
RM(b) = \frac{(1 - \rho)^2}{n} \left[ \frac{1 + r}{\rho(1 - \rho)} + \gamma_{2w} \right].
\]

(30)

It is thus seen that the estimator \(b\) is generally biased for both the structural and functional models. It vanishes for the structural model in two cases. One is when the error distributions are mesokurtic and the other is when the error distributions are not mesokurtic but satisfy the condition (25) with \(d = 1\). Comparing the relative mean squared errors and observing that \((2 + \gamma_{2w})\) cannot be negative, we find that the estimator \(b\) is more efficient when used for functional model in comparison to structural model.

**5. CONCLUSIONS**

We have considered the estimation of slope coefficient in a linear regression model when both the variables are subject to measurement errors.
Employing an ultrastructural framework and assuming the reliability ratio to be known, a consistent estimation procedure is described and its large sample asymptotic properties are studied when errors are not necessarily normally distributed. Our analysis has revealed some interesting results related to effect of departure from the normality of measurement errors on the efficiency properties of the consistent estimator. The performance of this estimator in two popular models, namely functional and structural models, is also studied.

Our investigations can be easily extended to the case of a model containing more than one explanatory variable on the lines of Fuller ([1], Section 2.2) and Fuller and Hidiroglou [7]. Further, it would be interesting to conduct some Monte Carlo studies and to apply bootstrap techniques to some real world data. These may shed some additional light on the performance properties of estimator. Employing the present consistent estimator, one can formulate predictors for the actual and average values of the study variable and explanatory variables. Analysing the properties of predictors will be an interesting exercise.

We have assumed that the true value of reliability ratio is known. Often this value may not be available. In such circumstances, it may be imperative to estimate it, for instance, using repeated observations or utilizing some independent sample or obtaining some guess. Every one of these propositions will alter the properties of estimator. To what extent will be an interesting question for investigation; see Stanley [8].

Acknowledgement—The second author gratefully acknowledges the financial support from the Council of Scientific and Industrial Research (CSIR) of India in the form of a Junior Research Fellowship.

REFERENCES


APPENDIX

Up to order $O(1)$, we have

$$n^{-1/2}E(h_x) = \rho(1 - d)$$  \hfill (A1)
$$E(h_x h_y) = \rho^2[(d^2 \gamma_{2w} - 2(1 - d)^2) - \rho(1 - \rho) \gamma_{2w}]$$  \hfill (A2)
$$E(h_x^2) = \rho(1 + r) \rho[1 + \rho] + \rho^2(\gamma_{2w} - 2) + 2(1 - d).$$  \hfill (A3)

Proof. It may be recalled that the three sets $(u_1, u_2, \ldots, u_n)$, $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_n)$ are stochastically independent. Further, the elements in each set are identically and independently distributed. Utilizing these properties, it is easy to see that

$$n^{-1/2}E(Z_{uw}) = -\sigma_u^2$$
$$n^{-1/2}E(Z_{uw}) = -\sigma_v^2$$

while the expected values of $Z_{uw}$, $Z_{uw}$, $Z_{uw}$, $Z_{uw}$, and $Z_{uw}$ are all zero.

Employing these results we obtain from eqn (15) the result (A1).

Similarly, dropping the terms with zero expected values, we find from eqns (14) and (15) the following result:

$$E(h_x h_y) = \frac{(1 - \rho)}{\sigma_u^2} E[(1 - \rho)Z_{uw} - \rho Z_{2w} - 2(1 - \rho)(Z_{vw} + Z_{2w}) + 4(1 - \rho)Z_{uw}].$$

It can be easily verified that

$$E(Z_{2w}) = \sigma_v^2 (2 + \gamma_{2w})$$
$$E(Z_{2w}) = \sigma_u^2 (2 + \gamma_{2w})$$
$$E(Z_{2w}) = \sigma_u^2 S_{uw}$$
$$E(Z_{2w}) = \sigma_u^2 S_{uw}$$
$$E(Z_{2w}) = \sigma_u^2 S_{uw}$$

up to order $O(1)$.

Using these expressions, we find the result (A2).

Equation (A3) can be derived in a similar manner.