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Shalabh \(^a\), Gaurav Garg \(^b\) & Neeraj Misra \(^a\)

\(^a\) Department of Mathematics & Statistics, Indian Institute of Technology Kanpur, Kanpur, India
\(^b\) Indian Institute of Management Lucknow, Lucknow, India

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Estimation of Regression Coefficients in a Restricted Measurement Error Model Using Instrumental Variables

SHALABH, GOURAV GARG, AND NEERAJ MISRA

Department of Mathematics & Statistics, Indian Institute of Technology Kanpur, Kanpur, India
Indian Institute of Management Lucknow, Lucknow, India

The use of instrumental variable approach is extended in measurement error models to the case when regression coefficients are subjected to exact linear restrictions. Some consistent estimators of regression coefficients are obtained which satisfy the restrictions also. We do not make any assumption the distribution of measurement errors and they need not to have necessarily a normal distribution. The asymptotic properties of the estimators are studied. A simulation study is simultaneously conducted to investigate the finite sample properties and compare the efficiencies of the proposed estimators.

Keywords Exact linear restrictions; Instrumental variables; Measurement errors; Ultrasubstructural model.

Mathematics Subject Classification Primary 62J05; Secondary 62H10, 62H12.

1. Introduction

The presence of measurement errors in observations disturbs the statistical inferences that are meant for no measurement error situations. In the context of linear regression model, the ordinary least squares estimator (OLSE) possesses minimum variance in the class of linear and unbiased estimators. When measurement errors enter into the observations, the same OLSE becomes not only biased but inconsistent also. The regression coefficients can be consistently estimated using some known additional information from outside the sample, for example, measurement error variances, reliability ratio, etc.; see Cheng and Van Ness (1999) and Fuller (1987) for more details. In multivariate measurement error models, the availability of covariance matrix of measurement errors or reliability matrix associated with explanatory variables are usually utilized to obtain the
consistent estimators of regression coefficients, see for example, Gleser (1992) and Shalabh (1998, 2003) for more details. Availability of such prior information is a big constraint in obtaining the consistent estimators of parameters; see (Kleeper and Leamer, 1984, p. 163).

Another method for estimating the regression coefficients consistently is the use of instrumental variables. In this approach, no assumption is made about the availability of any additional information about parameters of the model. Instead, it is assumed that some instrumental variables are available corresponding to explanatory variables. The IV estimation is quite popular in the literature and is studied from different perspectives. The IV estimation is discussed from decision theoretic approach in a simultaneous equation model in Chamberlain (2007), from nonparametric approach in Hall and Horowitz (2005), for grouped data in Bekker and van der Ploeg (2005), for panel data in Wooldridge (2005), for correlated instruments and disturbance in Iwata (1992), for binary regression in Stefanski and Buzas (1995), in nonlinear measurement error model by Amemiya (1990), Buzas (1997), Schennach (2007), etc. A complete bibliography of IV estimation is out of purview of this article.

In many applications, some prior information about regression coefficients is also available or some constrains are imposed on regression coefficients. Such information may arise from different sources like past experience or long association of experimenter with the experiment, similar kind of studies conducted in the past etc. Use of such prior information generally results in more efficient estimators. When such prior information or constraints are expressed in the form of exact linear restrictions then the restricted least squares estimator (RLSE) is used in a no measurement error linear regression model. This estimator is unbiased and more efficient than OLSE in the sense of covariance matrix when there is no measurement errors in the observations; see Toutenburg (1982) and Rao et al. (2008) for more details.

When observations are measurement error ridden, then both RLSE and OLSE become biased and inconsistent. The problem of consistent estimation of regression coefficients in measurement error models under constraints is discussed by Shalabh et al. (2007, 2009) using the covariance matrix of measurement errors associated with explanatory variables. When the covariance matrix of measurement errors is not available, their estimators can not be used. The problem of consistent estimation of regression coefficients which are subjected to some exact linear restrictions is considered in this article when no additional information is available. The instrumental variables approach is used to deal with the problem. This problem has not received much attention in literature. Giles (1982a,b) discussed such an issue in a no measurement error model when the linear restrictions on regression coefficients are exact and stochastic, respectively in nature. He derives an estimator and its asymptotic properties when the observations are free from measurement errors. What happens to the properties of estimator in the presence of measurement errors is not considered. The present work considers some more approaches and extends the IV estimation under exact linear restrictions.

Most of the literature related to measurement error models assumes the measurement errors to be normally distributed. When there is a departure from normality, the statistical conclusions become invalid and impious. Peixe et al. (2006) derived the mean squared error of IV estimator under an elliptical distribution of disturbances. Srivastava and Shalabh (1997a,b) and Shalabh (1998, 2003) reported
some results when the measurement errors are not necessarily normally distributed but without using the set up of exact restrictions and IV estimation. We assume only the existence and finiteness of first four moments of the distributions of measurement errors without associating any particular probability distribution in this article.

This article is organized as follows. Section 2 describes the multivariate ultrastructural model, instrumental variables and exact linear restrictions along with the various statistical assumptions required for the analysis. In Sec. 3, four different restricted estimators are obtained using instrumental variable approach. The asymptotic properties of the proposed estimators are studied in Sec. 4. The dominance of estimators over each other is studied under the criterion of Löwner ordering. The outcomes of simulation study are discussed in Sec. 5 followed by some concluding remarks presented in Sec. 6. Finally, Appendix presents the derivation of the results.

2. Model and Assumptions

Let \( \eta \) be the \( n \times 1 \) vector of \( n \) observations on true study variable \( \eta \) and let \( T := (\xi_1, \xi_2, \ldots, \xi_n)^\prime \), \( \xi_i = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{ip})^\prime \); \( i = 1, 2, \ldots, n \) be the \( n \times p \) matrix of \( n \) observations on \( p \times 1 \) true explanatory vector \( \xi \). We postulate the following relationship between \( \eta \) and \( T \):

\[
\eta = T\beta,
\]

where \( \beta \) is a \( p \times 1 \) vector of regression coefficients. Due to the presence of measurement errors, we can not observe \( \eta \) and \( T \). Instead, we observe \( y \) and \( X \) as

\[
y = \eta + \epsilon, \quad X = T + \Delta,
\]

where \( y \) is a \( n \times 1 \) vector of observed values on true study variable, \( X \) is a \( n \times p \) matrix of observed explanatory variable, \( \epsilon := (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^\prime \) is the vector of measurement errors associated with \( y \) and \( \Delta := (\delta_1, \delta_2, \ldots, \delta_n)^\prime \), \( \delta_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{ip})^\prime ; i = 1, 2, \ldots, n \) is the matrix of measurement errors involved in \( X \). We assume that \( \xi_i = \mu_i + \phi_i \), where \( \mu_i = (\mu_{i1}, \mu_{i2}, \ldots, \mu_{ip})^\prime \) are unknown parameters where \( \phi_i \) are independent and identically distributed (i.i.d.) random vectors with \( E(\phi_i) = 0 \). Thus, we can express,

\[
T = M + \Phi,
\]

where \( M := (\mu_1, \mu_2, \ldots, \mu_p)^\prime \), \( \mu_i = (\mu_{i1}, \mu_{i2}, \ldots, \mu_{ip})^\prime \) and \( \Phi := (\phi_1, \phi_2, \ldots, \phi_p)^\prime \), \( \phi_i = (\phi_{i1}, \phi_{i2}, \ldots, \phi_{ip})^\prime ; i = 1, 2, \ldots, n \). The set up (1)-(3) completes the specification of a multivariate ultrastructural model; see Dolby (1976). The ultrastructural model provides a general framework for the study of three interesting models, viz., functional and structural forms of measurement error model as well as classical regression model without measurement errors in a unified manner; see Shalabh et al. (2009).

Consider a \( p \times 1 \) random vector \( z \) of \( p \) instrumental variables. Let \( n \) observations on \( z \) be available as \( Z = (z_1, z_2, \ldots, z_n)^\prime \), \( z_i = (z_{i1}, z_{i2}, \ldots, z_{ip})^\prime , i = 1, 2, \ldots, n \). Thus, \( Z \) is the matrix of observations on instrumental variables. We
assume that, \( z_i = v_i + \psi_i \), where, \( v_i = (v_{i1}, v_{i2}, \ldots, v_{ip})' \) are unknown parameters and \( \psi_i = (\psi_{i1}, \psi_{i2}, \ldots, \psi_{ip})' \) are random vectors with \( E(\psi_i) = 0 \). In matrix notations,

\[
Z = N + \Psi,
\]

where \( N = (v_1, v_2, \ldots, v_n)' \) and \( \Psi = (\psi_1, \psi_2, \ldots, \psi_n)' \) are \( n \times p \) matrices. We also make the following assumptions.

A1. \( \epsilon_i, (i = 1, 2, \ldots, n) \) are i.i.d. random variables with mean 0 and variance \( \sigma_\epsilon^2 \).
A2. \( \delta_{ij}, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, p) \) are i.i.d. random variables with mean 0, variance \( \sigma_\delta^2 \) and finite fourth moment.
A3. \( \phi_{ij}, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, p) \) are i.i.d. random variables with mean 0, variance \( \sigma_\phi^2 \) and finite fourth moment.
A4. \( \psi_{ij}, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, p) \) are i.i.d. random variables with mean 0, variance \( \sigma_\psi^2 \) and finite fourth moment.
A5. \( \Phi, \Delta \) and \( \epsilon \) are stochastically independent.
A6. \( \Psi \) and \( (\epsilon, \Delta) \) are statistically independent.
A7. \( E(\psi_i \psi_j') = \begin{cases} \Sigma_{\psi\psi}, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \) for \( i, j = 1, 2, \ldots, n \).
A8. \( E(\psi_i^2 \phi_{ik}) = \gamma_{\psi\phi} < \infty \) for \( i = 1, 2, \ldots, n, j, k = 1, 2, \ldots, p \).
A9. \( Z'X \) is non singular with probability one.
A10. \( \lim_{n \to \infty} \mu_n = \sigma_\mu, \lim_{n \to \infty} v_n = \sigma_v \).

Assumption A10 implies that

\[
\lim_{n \to \infty} n^{-1}N'N = \sigma, \sigma' \quad \lim_{n \to \infty} n^{-1}N'M = \sigma, \sigma'_M \quad \text{and} \quad \lim_{n \to \infty} n^{-1}M'M = \sigma, \sigma'_M. \tag{5}
\]

Assumption A10 is required to derive asymptotic distribution of the estimators. They also rule out the presence of any trend in observations; see Schneeweiss (1991). Let \( \Sigma_1 := \sigma, \sigma'_M + \sigma^2_\psi I_p \), \( \Sigma_2 := \sigma, \sigma'_M + \sigma^2_\epsilon I_p \), and \( \Sigma_3 := \sigma, \sigma'_M + \Sigma_{\psi\epsilon} \). Matrices \( \Sigma_1 \) and \( \Sigma_2 \) are non singular when \( \sigma^2_\psi + \sigma^2_\epsilon > 0 \) and \( \sigma^2_\psi > 0 \), respectively. We assume that matrix \( \Sigma_3 \) is non singular.

We present some results in the following Lemma. These results are used in further analysis.

**Lemma 2.1.** Under the assumptions A1–A10, (i) \( \quad \lim_{n \to \infty} n^{-1}X'X = \Sigma_1 \), (ii) \( \quad \lim_{n \to \infty} n^{-1}X'y = (\Sigma_1 - \sigma^2_\psi I_p)\beta \), (iii) \( \quad \lim_{n \to \infty} n^{-1}Z'Z = \Sigma_2 \), (iv) \( \quad \lim_{n \to \infty} n^{-1}Z'(\epsilon - \Delta \beta) = 0 \), and (v) \( \quad \lim_{n \to \infty} n^{-1}Z'X = \Sigma_3 \), where \( \lim(\cdot) \) denotes convergence in probability.

Further, we assume the availability of prior information on regression coefficients or some constraints binding the regression coefficients in the form of exact linear restrictions. The regression coefficients \( \beta \) are subjected to \( J(< p) \) independent exact linear restrictions given by

\[
r = R\beta, \tag{6}
\]

where \( r \) is a \( J \times 1 \) known vector and \( R \) is a \( J \times p \) known matrix such that \( \text{rank}(R) = J \). We assume that \( J \) is strictly less than \( p \), because when \( J \geq p \), unknown \( \beta \)
can be obtained from the restrictions only and the problem becomes trivial and meaningless.

3. Instrumental Variable Estimation of Regression Coefficients

The model (1)–(2) can be expressed as

\[ y = X\beta + (\epsilon - \Delta\beta), \] (7)

where \( X \) is regressor matrix, \( (\epsilon - \Delta\beta) \) is composite disturbance term. The model (7) resembles with usual linear regression model. From the assumptions A1, A2, and A5, we have \( E(X'(\epsilon - \Delta\beta)) = -n\sigma^2_\delta \beta \neq 0 \), in general. It is zero when either \( \sigma^2_\delta = 0 \) or \( \beta = 0 \), which are not the interesting cases. Thus, explanatory variables and disturbance term are correlated which violates the assumption of classical regression. This makes the usual estimators to loose their optimal properties. The ordinary least squares estimator (OLSE) \( b \) of \( \beta \) and the restricted least squares estimator (RLSE) \( b_R \) of \( \beta \) under restrictions (6) in a classical regression model without measurement errors are given by

\[ b = (X'X)^{-1}X'y \] (8)

and

\[ b_R = b + (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(r - Rb), \] (9)

respectively. The RLSE \( b_R \) can also be expressed as

\[ b_R = b - (I_p - f_R(X'X))(b - \beta), \] (10)

where the function \( f_R : \mathbb{R}^{p \times p} \rightarrow \mathbb{R}^{p \times p} \) is defined as

\[ f_R(U) = I_p - U^{-1}R'(RU^{-1}R')^{-1}R, \quad U \in \mathbb{R}^{p \times p}. \] (11)

From Lemma 2.1(i) and (ii), we have

\[ \text{plim}_{n \to \infty} (b - \beta) = -\sigma^2_\delta \Sigma^{-1}_1 \beta \neq 0, \] (12)

\[ \text{plim}_{n \to \infty} (b_R - \beta) = -\sigma^2_\delta \Sigma^{-1}_1 (I_p - R'(RU^{-1}R')^{-1}R\Sigma^{-1}_1)\beta \neq 0. \] (13)

Thus, both \( b \) and \( b_R \) are inconsistent for \( \beta \) under the model (1)–(3). It is clear from (12) and (13), that OLSE and RLSE become consistent for estimating \( \beta \) when \( \sigma^2_\delta = 0 \). We have already assumed that \( E(\delta_i) = 0 \). Therefore, \( \text{Var}(\delta_i) = \sigma^2_\delta = 0 \) makes the measurement errors \( \delta_i \) identically zero and the model converts to a classical multiple linear regression model with no measurement errors. The OLSE \( b \) does not satisfy the restrictions (6), that is, \( Rb \neq r \) while RLSE \( b_R \) satisfies the given restrictions, that is, \( Rb_R = r \).

In order to estimate \( \beta \) consistently under the model (1)–(3), we make use of instrumental variable observations \( Z \). To achieve this, the weighted score
function \((y - X\beta)'Z(Z'Z)^{-1}Z'(y - X\beta)\) is minimized with respect to \(\beta\). This yields the estimator

\[
\hat{\beta}^{(1)}_IV = (Z'X)^{-1}Z'y;
\]  

(14)

see (Fuller, 1987, p. 150) and (Cheng and Van Ness, 1999, p. 145) for more details. Using Lemma 2.1(iv) and (v), we have

\[
\text{plim}_{n \to \infty} \hat{\beta}^{(1)}_IV = \beta.
\]  

(15)

However, \(\hat{\beta}^{(1)}_IV\) does not satisfy the restrictions (6), that is, \(R\hat{\beta}^{(1)}_IV \neq r\). For some specific value of \(\hat{\beta}^{(1)}_IV\), the relation \(R\hat{\beta}^{(1)}_IV = r\) could hold, but it does not hold in general. Therefore our objective is to obtain such estimators of \(\beta\) which are consistent as well as satisfy the linear restrictions (6). We now obtain three such estimators of \(\beta\) under the measurement error model (1)–(4).

In the first approach, we minimize the weighted score function \([(y - X\hat{\beta}^{(1)}_IV)'(y - X\hat{\beta}^{(1)}_IV)]^{-1} \times (\hat{\beta}^{(1)}_IV - \beta)'S_{xz}(\hat{\beta}^{(1)}_IV - \beta)\) with respect to \(\beta\) subject to the restrictions \(R\beta = r\), where \(S_{xz} = (X'Z)(Z'Z)^{-1}(Z'X)\). The resulting estimator is

\[
\hat{\beta}^{(2)}_IV = \hat{\beta}^{(1)}_IV + S_{xz}^{-1}R[RS_{xz}^{-1}R']^{-1}(r - R\hat{\beta}^{(1)}_IV).
\]  

(16)

The estimator \(\hat{\beta}^{(2)}_IV\) can also be equivalently expressed as

\[
\hat{\beta}^{(2)}_IV = \hat{\beta}^{(1)}_IV - [I_p - f_R(S_{xz})](\hat{\beta}^{(1)}_IV - \beta).
\]  

(17)

The motivation of using the weight \([(y - X\hat{\beta}^{(1)}_IV)'(y - X\hat{\beta}^{(1)}_IV)]^{-1}S_{xz}\) is as follows. In Theorem 4.1, given later, we show that the asymptotic distribution of \(\sqrt{n}(\hat{\beta}^{(1)}_IV - \beta)\) is \(p\)-variate normal with mean vector \(\theta\) and covariance matrix \(\Omega_{IV}^{(1)} = (\sigma_\epsilon^2 + \sigma_\beta^2\beta\beta'S_{z})^{-1}S_{z}(S_{z})^{-1}\). We observe that \((y - X\hat{\beta}^{(1)}_IV)'(y - X\hat{\beta}^{(1)}_IV)S_{xz}^{-1}\) is a feasible estimate of \(\Omega_{IV}^{(1)}\).

Using Lemma 2.1, we get \(\text{plim}_{n \to \infty} f_R(S_{xz}) = f_R(\Sigma_1 \Sigma_2^{-1} \Sigma_3)\). Thus, using (15), we get \(\text{plim}_{n \to \infty} \hat{\beta}^{(2)}_IV = \beta\). Since \(f_R(S_{xz}) = 0\), it follows that \(R\hat{\beta}^{(2)}_IV = r\).

Interestingly enough, the estimator \(\hat{\beta}^{(2)}_IV\) given in (17) can also be obtained by minimizing the score function \((y - X\beta)'Z(Z'Z)^{-1}Z'(y - X\beta)\) with respect to \(\beta\) subject to the restrictions \(R\beta = r\). Using this approach, Giles (1982a) obtained a similar estimator in a linear regression model without measurement errors.

Another consistent estimator of \(\beta\) is obtained by replacing the inconsistent estimator \(\hat{\beta}^{(1)}_IV\) in \(\hat{\beta}^{(2)}_IV\) by the consistent estimator \(\hat{\beta}^{(3)}_IV\) in \(\hat{b}_R\). The obtained estimator is

\[
\hat{\beta}^{(3)}_IV = \hat{\beta}^{(1)}_IV + (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(r - R\hat{\beta}^{(1)}_IV),
\]  

(18)

which can also be expressed as

\[
\hat{\beta}^{(3)}_IV = \hat{\beta}^{(1)}_IV - [I_p - f_R(X'X)](\hat{\beta}^{(1)}_IV - \beta).
\]  

(19)
Using Lemma 2.1 (i), we get \( \text{plim}_{n \to \infty} f_R(X'X) = f_R(\Sigma_1) \). Therefore, using (15), we have \( \text{plim}_{n \to \infty} \hat{\beta}_R^{(3)} = \beta \). Since \( Rf_R(X'X) = 0 \), we have \( R\hat{\beta}_R^{(3)} = r \). Thus, \( \hat{\beta}_R^{(3)} \) is consistent for \( \beta \) as well as satisfies the linear restrictions (6).

It is observed that \( \hat{\beta}_R^{(3)} \) can also be obtained by minimizing the weighted error sum of squares \( (\hat{\beta}_R^{(3)} - \beta)'(X'X)(\hat{\beta}_R^{(3)} - \beta) \) with respect to \( \beta \) subject to the restrictions \( R\beta = r \).

To obtain another restricted consistent estimator of \( \beta \), we minimize the weighted error sum of squares \( (\hat{\beta}_R^{(1)} - \beta)'(Z'Z)(\hat{\beta}_R^{(1)} - \beta) \) with respect to \( \beta \) subject to the restrictions \( R\beta = r \). This yields the following estimator:

\[
\hat{\beta}_R^{(4)} = \hat{\beta}_R^{(1)} + (Z'Z)^{-1}R'[R(Z'Z)^{-1}R]'^{-1}(r - R\hat{\beta}_R^{(1)}),
\]

which can also be expressed as

\[
\hat{\beta}_R^{(4)} = \hat{\beta}_R^{(1)} - (I_p - f_R(Z'Z))(\hat{\beta}_R^{(1)} - \beta).
\]

Using Lemma 2.1(iii), we get \( \text{plim}_{n \to \infty} f_R(Z'Z) = f_R(\Sigma_2) \). Therefore, using (15), we find that \( \hat{\beta}_R^{(4)} \) is a consistent estimator of \( \beta \). Since \( Rf_R(Z'Z) = 0 \), \( \hat{\beta}_R^{(4)} \) satisfies the given linear restrictions (6), that is, \( R\hat{\beta}_R^{(4)} = r \).

In the next approach, we use weight matrix \( Z'X \) and minimize the weighted error sum of squares \( (\hat{\beta}_R^{(1)} - \beta)'(Z'X)(\hat{\beta}_R^{(1)} - \beta) \) with respect to \( \beta \) subject to the linear restrictions \( R\beta = r \). Proceeding as earlier, we obtain the following estimator of \( \beta \):

\[
\hat{\beta}_R^{(5)} = \hat{\beta}_R^{(1)} + (Z'X)^{-1}R'[R(Z'X)^{-1}R]'^{-1}(r - R\hat{\beta}_R^{(1)}),
\]

which can be presented in the following form:

\[
\hat{\beta}_R^{(5)} = \hat{\beta}_R^{(1)} - (I_p - f_R(Z'X))(\hat{\beta}_R^{(1)} - \beta).
\]

Using Lemma 2.1(v), we have \( \text{plim}_{n \to \infty} f_R(Z'X) = f_R(\Sigma_4) \). Therefore, since \( \hat{\beta}_R^{(1)} \) is consistent for \( \beta \), \( \hat{\beta}_R^{(5)} \) is also a consistent estimator of \( \beta \). It is also clear that the estimator \( \hat{\beta}_R^{(5)} \) satisfies the linear restrictions (6).

Thus, we have four consistent estimators \( \hat{\beta}_R^{(l)}, l = 2, 3, 4, 5 \) of \( \beta \) which also satisfy the restrictions (6).

4. **Asymptotic Properties**

It is difficult to obtain the exact sample properties of the estimators \( \hat{\beta}_R^{(l)}, l = 1, 2, 3, 4, 5 \). Even if they are derived, their structure may be too complicated to provide any clear conclusion. Therefore, we study the asymptotic properties of these estimators. Following theorem presents the asymptotic distributions of the estimators \( \hat{\beta}_R^{(1)}, \hat{\beta}_R^{(2)}, \hat{\beta}_R^{(3)}, \hat{\beta}_R^{(4)}, \) and \( \hat{\beta}_R^{(5)} \).

**Theorem 4.1.** The asymptotic distribution of \( \sqrt{n}(\hat{\beta}_R^{(l)} - \beta) \) \( (l = 1, 2, 3, 4, 5) \) is \( p \)-variate normal with mean vector \( \mathbf{0} \) and covariance matrix \( \Omega_R^{(l)} = (\sigma_1^2 + \sigma_2^2 \beta \beta)'D_\beta^2 \Sigma_4 D_\beta^2 \).
where
\[
D_1^* = \Sigma_1^{-1}, \quad D_2^* = f_k(\Sigma_1')\Sigma_2^{-1}\Sigma_3, \quad D_3^* = f_k(\Sigma_1')\Sigma_2^{-1},
\]
\[
D_4^* = f_k(\Sigma_2')\Sigma_3^{-1}, \quad D_5^* = f_k(\Sigma_2')\Sigma_3^{-1}.
\]

(24)

Proof. See the Appendix.

In order to observe the gain in efficiency in the restricted estimators \(\hat{\beta}_N^{(l)}\), \(l = 2, 3, 4, 5\), over unrestricted estimator \(\hat{\beta}_N^{(1)}\), we compare the covariance matrices of their asymptotic distributions and note that

\[
\Omega_N^{(1)} - \Omega_N^{(4)} = (\sigma_\epsilon^2 + \sigma_\beta^2 \beta)(\Sigma_1'/\Sigma_2'^{-1}\Sigma_3)^{-1}R[R(\Sigma_1'/\Sigma_2'^{-1}\Sigma_3)^{-1}R]^{-1}R(\Sigma_1'/\Sigma_2'^{-1}\Sigma_3)^{-1},
\]

which is a non negative definite matrix. Therefore, \(\hat{\beta}_N^{(4)}\) is more efficient than \(\hat{\beta}_N^{(1)}\) in the sense of Löwner ordering of the covariance matrices of their asymptotic distributions. No clear analytic condition for the dominance of restricted estimators \(\hat{\beta}_N^{(l)}\), \(l = 3, 4, 5\), over unrestricted estimator \(\hat{\beta}_N^{(1)}\) is found under of Löwner ordering of the covariance matrices of their asymptotic distributions. The dominance of \(\hat{\beta}_N^{(l)}\), \(\hat{\beta}_N^{(2)}\), \(\hat{\beta}_N^{(3)}\), \(\hat{\beta}_N^{(4)}\), and \(\hat{\beta}_N^{(5)}\) over each other is studied in the sense of stochastic closeness to the true parameter \(\beta\). We provide sufficient conditions for the dominance of these estimators over each other in this sense. The results are the consequences of Theorem 4.1 and stated in the following Corollary.

**Corollary 4.1.** If \((D_k^*/D_k^* - D_k^*/D_k^*)\) is a positive semi-definite matrix then for \(k, l = 1, 2, 3, 4, 5\); \(k \neq l\) and for every \(x \geq 0\),

\[
\lim_{n \to \infty} P[\|\sqrt{n}(\hat{\beta}_N^{(k)} - \beta)\| \leq x] \leq \lim_{n \to \infty} P[\|\sqrt{n}(\hat{\beta}_N^{(l)} - \beta)\| \leq x],
\]

(25)

where \(D_1^*, D_2^*, D_3^*, D_4^*, \) and \(D_5^*\) are defined in (24) and \(\|\cdot\|\) is the usual Euclidian norm.

The stochastic dominance implies universal dominance (see Hwang, 1985). Therefore, condition (25) implies that under certain regulatory conditions allowing the change of limit and expectations, for every non-decreasing function \(L(\cdot)\),

\[
E[L(\|\sqrt{n}(\hat{\beta}_N^{(k)} - \beta)\|)] \geq E[L(\|\sqrt{n}(\hat{\beta}_N^{(l)} - \beta)\|)],
\]

as \(n \to \infty\), for \(k, l = 1, 2, 3, 4, 5\); \(k \neq l\).

It is clear from Theorem 4.1 that the covariance matrices of the asymptotic distributions of these estimators are not affected by the nonnormality effects of the distributions of measurement errors, true regressors and the instrumental variables. This effect of nonnormality does not precipitate in large samples properties and may be present in small sample properties. We make an attempt to study the nonnormality effect through Monte Carlo simulation experiments in the next section.

### 5. Simulation Study

We conducted a simulation study to study the performance of estimators in small samples. Different values of \(\epsilon_i, \phi_{ij}, \delta_{ij}, R, \) and \(\beta\) are adopted and the observations on
The EBLs of all the estimators have normal distribution as such that the correlation between the estimators are given in Tables 4–6. The EBLs of the estimators are given in Tables 1–3 and EMSEs of mean squared error matrix (EMSEM), and empirical mean squared error (EMSE) of these estimators. To save space, we present only a few simulation outcomes in the Tables 1–6. The EBLs of the estimators are given in Tables 1–3 and EMSEs of the estimators are given in Tables 4–6.

The EBLs of the estimators $\hat{\beta}_{IV}^{(l)}$, $l = 1, 2, 3, 4, 5$, tend towards zero as sample size increases under all distributional assumptions and all different combinations of $\sigma_\epsilon^2$, $\sigma_\phi^2$, and $\sigma_\delta^2$. The EBLs of $\hat{\beta}_{IV}^{(4)}$ and $\hat{\beta}_{IV}^{(5)}$ are similar and they are smaller than the EBLs of $\hat{\beta}_{IV}^{(1)}$, $\hat{\beta}_{IV}^{(2)}$, and $\hat{\beta}_{IV}^{(3)}$ under all settings. The EBL of $\hat{\beta}_{IV}^{(3)}$ is less than the EBL of $\hat{\beta}_{IV}^{(2)}$. The estimators $\hat{\beta}_{IV}^{(2)}$ and $\hat{\beta}_{IV}^{(3)}$ do not have clear cut dominance over the estimator $\hat{\beta}_{IV}^{(1)}$ in the sense of EBL. Under all distributional assumptions, the EBLs of all the estimators $\hat{\beta}_{IV}^{(l)}$, $l = 1, 2, 3, 4$, increase as $\sigma_\delta^2$ is increased and decrease as $\sigma_\phi^2$ is increased. The effect of $\sigma_\epsilon^2$ on EBLs is not significant. The EBLs of the

<table>
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<tr>
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<tr>
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<td>$\sigma_\phi^2$</td>
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</table>
| $\epsilon_i, \phi_{ij}, \delta_{ij}$ are generated using normal, Student’s $t$, and Gamma distributions. These distributions are adopted to study the effect of non normality in terms of skewness and kurtosis. The observations $\psi_{ij}$ ($i = 1, 2, \ldots, n, j = 1, 2, \ldots, 5$) are generated such that the correlation between $\psi_{ij}$ and $\phi_{ij}$ is 0.95. The matrices $M$ and $N$ are kept fixed throughout the simulation. We compute the estimators $\hat{\beta}_{IV}^{(l)}$, $l = 1, 2, 3, 4, 5$, for 100,000 times repeatedly and obtain empirical bias length (EBL), empirical mean squared error matrix (EMSEM), and empirical mean squared error (EMSE) of these estimators. To save space, we present only a few simulation outcomes in the Tables 1–6. The EBLs of the estimators are given in Tables 1–3 and EMSEs of the estimators are given in Tables 4–6. The EBLs of the estimators $\hat{\beta}_{IV}^{(l)}$, $l = 1, 2, 3, 4, 5$, tend towards zero as sample size increases under all distributional assumptions and all different combinations of $\sigma_\epsilon^2$, $\sigma_\phi^2$, and $\sigma_\delta^2$. The EBLs of $\hat{\beta}_{IV}^{(4)}$ and $\hat{\beta}_{IV}^{(5)}$ are similar and they are smaller than the EBLs of $\hat{\beta}_{IV}^{(1)}$, $\hat{\beta}_{IV}^{(2)}$, and $\hat{\beta}_{IV}^{(3)}$ under all settings. The EBL of $\hat{\beta}_{IV}^{(3)}$ is less than the EBL of $\hat{\beta}_{IV}^{(2)}$. The estimators $\hat{\beta}_{IV}^{(2)}$ and $\hat{\beta}_{IV}^{(3)}$ do not have clear cut dominance over the estimator $\hat{\beta}_{IV}^{(1)}$ in the sense of EBL. Under all distributional assumptions, the EBLs of all the estimators $\hat{\beta}_{IV}^{(l)}$, $l = 1, 2, 3, 4$, increase as $\sigma_\delta^2$ is increased and decrease as $\sigma_\phi^2$ is increased. The effect of $\sigma_\epsilon^2$ on EBLs is not significant. The EBLs of the
Estimation of Regression Coefficients

Table 2
Empirical bias length of $\hat{\beta}^{(l)}_W$, $l = 1, 2, 3, 4, 5$, when $(e, \phi, \delta)$ have $t$ distribution with 12 degrees of freedom

<table>
<thead>
<tr>
<th>$n = 22$</th>
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<tbody>
<tr>
<td>$2.200$</td>
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<tr>
<td>$0.007$</td>
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</table>

estimators are not significantly affected by the effect of departure from normality of the distributions of different errors.

Under all distributional assumptions, the EMSE of $\hat{\beta}^{(l)}_W$, $(l = 1, 2, 3, 4, 5)$ decrease as sample size increases. From the EMSEM of these estimators, we observe that $\text{EMSEM}(\hat{\beta}^{(2)}_W) \leq \text{EMSEM}(\hat{\beta}^{(3)}_W) \leq \text{EMSEM}(\hat{\beta}^{(4)}_W) \leq \text{EMSEM}(\hat{\beta}^{(5)}_W)$. Thus, we observe that the unrestricted estimator $\hat{\beta}^{(3)}_W$ is least efficient in the sense of Löwner ordering of EMSEMs and $\hat{\beta}^{(5)}_W$ is the most efficient under the same criterion. Under all distributional assumptions, the EMSE of all the estimators $\hat{\beta}^{(l)}_W$, $(l = 1, 2, 3, 4, 5)$ increase as $\sigma_3^2$ is increased and decrease as $\sigma_3^2$ is increased. The effect of $\sigma_3^2$ on EMSE is not significant. The values of EMSE under different error distributions are not the same. This reflects that the departure from nonnormality of error distributions do affect the EMSEs. But the direction of this effect is not clear.

Thus, on the basis of simulation study, we conclude that no single estimator dominates the other estimators among $\hat{\beta}^{(l)}_W$, $(l = 1, 2, 3, 4, 5)$, in the sense of EBL. However, $\hat{\beta}^{(2)}_W$ dominates $\hat{\beta}^{(l)}_W$, $(l = 1, 3, 4, 5)$, in terms of Löwner ordering of EMSEMs. Thus, $\hat{\beta}^{(2)}_W$ is recommended for use. The information of exact linear restrictions $r = R\beta$ results in gain in efficiency of these estimators as the EMSE of
6. Concluding Remarks

We considered the problem of consistent estimation of regression coefficients when measurement errors are present in the data and regression coefficients are subjected to exact linear restrictions. The method of IV estimation is extended and four estimators are proposed which incorporate the restrictions in the IV estimation. These estimators are consistent under the influence of measurement errors in the data and satisfy the restrictions too. The asymptotic distributions and properties of these estimators are derived. The estimators are compared in the sense of stochastic closeness to the true parameter. We infer that the incorporation of prior information (6) in the proposed IV estimators leads to more efficient estimators under some conditions except in one case of $\hat{\beta}_{IV}^{(1)}$ which has a uniform superiority in the sense of Löwner ordering of covariance matrices of their asymptotic distributions.
Moreover, when we compare the simulated mean squared errors, we observe clearly that the incorporation of prior information always leads to more efficient estimators. Simulation study also recommends that \( \hat{\beta}^{(2)} \) is better to use over the other estimators. The simulation results confirm that departure from normality affects the finite sample properties of the proposed estimators.

In this approach of instrumental variables, the estimators are the functions of instrumental variable observations. Therefore, the properties of the estimators depend on the properties of the instrumental variables. When the instrumental variables are not available separately, they can be generated using the observed explanatory variables. Some methods for the generation of instrumental variables, including ranking method and grouping method, are discussed in Cheng and Van Ness (1999) and Fuller (1987).

## Appendix

First we state some lemmas which are used in the proof of Theorem 4.1.

### Lemma A.1 Central Limit Theorem.

Let \( V_n = \sum_{i=1}^{n} C_i \omega_i \), where \( \omega_1, \omega_2, \ldots, \omega_n \) are \( p \times 1 \) independent and identically distributed random vectors with \( E(\omega_i) = 0 \) and...
(iii) (Malinvaud, 1966, p. 212).

\( q \) defined in the previous lemma.

- Multivariate normal distribution with mean vector \( \bar{SC} \) and covariance matrix \( \phi \).

- Under the assumptions A1–A6, we have:

  1. \( E(Z \epsilon \epsilon' Z) = \sigma_1^2 N' N + n \sigma_1^2 \phi \delta_{\beta}' I_p \);  
  2. \( E(Z \Delta \beta \Delta' Z) = \sigma_2^2 \phi \beta' N' N + n \sigma_2^2 \phi \delta_{\beta}' I_p \);  
  3. \( E(g) = 0 \); and  
  4. \( E(\epsilon) = (\sigma_1^2 + \sigma_2^2 \phi \delta_{\beta}' I_p)(n^{-1} N' N + \sigma_2^2 I_p) \).

This lemma can be proved using the distributional properties of \( \epsilon, \Delta, \Phi, \) and \( \Psi \).

**Lemma A.2.** Let \( g := n^{-\frac{1}{2}} Z'(\epsilon - \Delta \beta) \). Under the assumptions A1–A6, we have:

- \( E(Z \epsilon \epsilon' Z) = \sigma_1^2 N' N + n \sigma_1^2 \phi \delta_{\beta}' I_p \);  
- \( E(Z \Delta \beta \Delta' Z) = \sigma_2^2 \phi \beta' N' N + n \sigma_2^2 \phi \delta_{\beta}' I_p \);  
- \( E(g) = 0 \); and  
- \( E(\epsilon) = (\sigma_1^2 + \sigma_2^2 \phi \delta_{\beta}' I_p)(n^{-1} N' N + \sigma_2^2 I_p) \).

**Lemma A.3.** As \( n \to \infty \), \( g \overset{d}{\to} N_p(0, (\sigma_1^2 + \sigma_2^2 \phi \delta_{\beta}' I_p) \Sigma_2) \), where \( N_p(\mu, \Sigma) \) denotes \( p \)-variate normal distribution with mean vector \( \mu \) and covariance matrix \( \Sigma \). Vector \( g \) is defined in the previous lemma.

**Table 5**

Empirical mean squared error of \( \hat{h}_h^{(l)} \), \( l = 1, 2, 3, 4, 5 \), when \( (\epsilon, \phi, \delta) \) have \( t \) distribution with 12 degrees of freedom

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<thead>
<tr>
<th>( n = 22 )</th>
<th>( n = 48 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2.200 )</td>
<td>( 2.000 )</td>
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<tr>
<td>( 1.100 )</td>
<td>( 5.000 )</td>
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<tr>
<td>( \beta \to 3.000 )</td>
<td>( 5.000 )</td>
</tr>
<tr>
<td>( 4.200 )</td>
<td>( 5.000 )</td>
</tr>
<tr>
<td>( 2.500 )</td>
<td>( 5.000 )</td>
</tr>
</tbody>
</table>

\( C_1, C_2, \ldots, C_m \) are \( q \times p \) non stochastic matrices. Suppose that \( \lim_{n \to \infty} \text{cov}(V_n) = \Omega \), where \( |(\Omega)_{ij}| < \infty \) and \( \Omega \) is positive definite. If there exists a function \( \phi(n) \) such that \( \lim_{n \to \infty} \phi(n) = \infty \) and if the elements of \( \phi(n)C_m \) are bounded, then \( V_n \) has a limiting \( q \)-variate normal distribution with mean vector \( \hat{0} \) and covariance matrix \( \phi \); see (Malinvaud, 1966, p. 212).

C_1, C_2, \ldots, C_m are q x p non stochastic matrices. Suppose that \( \lim_{n \to \infty} \text{cov}(V_n) = \Omega \), where \( |(\Omega)_{ij}| < \infty \) and \( \Omega \) is positive definite. If there exists a function \( \phi(n) \) such that \( \lim_{n \to \infty} \phi(n) = \infty \) and if the elements of \( \phi(n)C_m \) are bounded, then \( V_n \) has a limiting \( q \)-variate normal distribution with mean vector \( \hat{0} \) and covariance matrix \( \phi \); see (Malinvaud, 1966, p. 212).
Proof of Theorem 4.1. Using (14), (1)–(2), and (3), we get
\[
\sqrt{n}(\hat{\beta}^{(1)} - \beta) = \sqrt{n}(Z'X)^{-1}Z'(\epsilon - \Delta \beta) = \{n(Z'X)^{-1}\}g.
\]  
(26)

Now, using Lemma 2.1(v) and Lemma A.3, we obtain, as \( n \to \infty \),
\[
\sqrt{n}(\hat{\beta}^{(3)} - \beta) \xrightarrow{d} N_p(0, (\sigma_e^2 + \sigma_\phi^2 \beta^2) \Sigma_3^{-1} \Sigma_2 (\Sigma_1')^{-1}).
\]  
(27)

From (17), we get \( \sqrt{n}(\hat{\beta}^{(2)} - \beta) = f_R(S_{xz}) (\sqrt{n}(\hat{\beta}^{(1)} - \beta)) \), where \( S_{xz} = X'Z(Z'Z)^{-1}Z'X \). Using (27) and \text{plim} \( f_R(S_{xz}) \), we obtain, as \( n \to \infty \),
\[
\sqrt{n}(\hat{\beta}^{(2)} - \beta) \xrightarrow{d} N_p(0, (\sigma_e^2 + \sigma_\phi^2 \beta^2) f_R(\Sigma_3^{-1} \Sigma_2^{-1} \Sigma_1^{-1}) (\Sigma_2)^{-1} f_R(\Sigma_1' \Sigma_2^{-1} \Sigma_1')).
\]  
(28)

where \( f_R(\cdot) \) denotes the transpose of matrix \( f_R(\cdot) \).
From (19), we have $\sqrt{n}(\hat{\beta}_V^{(3)} - \beta) = f_R(X'X)[\sqrt{n}(\hat{\beta}_V^{(1)} - \beta)]$. Using (27) and $\text{plim } f_R(X'X)$, we obtain, as $n \to \infty$,

$$\sqrt{n}(\hat{\beta}_V^{(3)} - \beta) \xrightarrow{d} N_p(0, (\sigma^2 + \sigma^2_{\beta}\beta) f_R(\Sigma_1) (\Sigma_2) - 1 \Sigma_2 (\Sigma_3) - 1 f_R(\Sigma_1)).$$  \hfill (29)

From (21), we have $\sqrt{n}(\hat{\beta}_V^{(4)} - \beta) = f_R(Z'Z)[\sqrt{n}(\hat{\beta}_V^{(1)} - \beta)]$. From (27) and $\text{plim } f_R(Z'Z)$, we get, as $n \to \infty$,

$$\sqrt{n}(\hat{\beta}_V^{(4)} - \beta) \xrightarrow{d} N_p(0, (\sigma^2 + \sigma^2_{\beta}\beta) f_R(\Sigma_2) (\Sigma_3) - 1 \Sigma_2 (\Sigma_3) - 1 f_R(\Sigma_2)).$$  \hfill (30)

Using (23), we get $\sqrt{n}(\hat{\beta}_V^{(5)} - \beta) = f_R(Z'X)[\sqrt{n}(\hat{\beta}_V^{(1)} - \beta)]$. Using (27) and $\text{plim } f_R(Z'X)$, we get, as $n \to \infty$,

$$\sqrt{n}(\hat{\beta}_V^{(5)} - \beta) \xrightarrow{d} N_p(0, (\sigma^2 + \sigma^2_{\beta}\beta) f_R(\Sigma_1) (\Sigma_3) - 1 \Sigma_2 (\Sigma_3) - 1 f_R(\Sigma_2)).$$  \hfill (31)

The theorem is followed from (27), (28), (29), (30), and (31).

Acknowledgment

The authors are thankful to the referee for valuable comments which improved exposition of the article.

References


