Chapter 8
Logistic and Poisson Regression Models

Logistic Regression Models:

In the linear regression model $X\beta + \varepsilon$, there are two types of variables – explanatory variables $X_1, X_2, .., X_k$ and study variable y. These variables can be measured on a continuous scale and treated as indicator variables. When the explanatory variables are qualitative, their values are represented as indicator variables, and dummy variable models are used.

When the study variable is qualitative, its values can be represented by an indicator variable that takes only two possible values, 0 and 1. In such a case, the logistic regression is used. For example, y can denotes values like success or failure, yes or no, or like or dislike, which can be represented by two values: 0 and 1.

Consider the model

$$\begin{aligned} y_i &= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_k x_{ik} + \varepsilon_i \\ &= x_i' \beta + \varepsilon_i, \ i = 1, 2, ..., n \end{aligned}$$

where $x_i' = [1, x_{i1}, x_{i2}, ..., x_{ik}], \ \beta' = [\beta_0, \beta_1, \beta_2, ..., \beta_k]$.

The study variable takes two values as $y_i = 0$ or 1. Assume that y_i follows a Bernoulli distribution with a parameter π_i, so its probability distribution is

$$y_i = \begin{cases} 1 & \text{with } P(y_i = 1) = \pi_i \\ 0 & \text{with } P(y_i = 0) = 1 - \pi_i. \end{cases}$$

Assuming $E(\varepsilon_i) = 0$,

$$E(y_i) = 1.\pi_i + 0.(1 - \pi_i) = \pi_i.$$

From the model $y_i = x_i'\beta + \varepsilon_i$, we have

$$\begin{aligned} & E(y_i) = x_i'\beta \\ \Rightarrow & E(y_i) = x_i'\beta = \pi_i \\ \Rightarrow & E(y_i) = P(y_i = 1). \end{aligned}$$

Thus response function $E(y_i)$ is simply the probability that $y_i = 1$.

Note that $\varepsilon_i = y_i - x_i'\beta$, so

- when $y_i = 1$, then $\varepsilon_i = 1 - x_i'\beta$
- $y_i = 0$, then $\varepsilon_i = -x_i'\beta$.

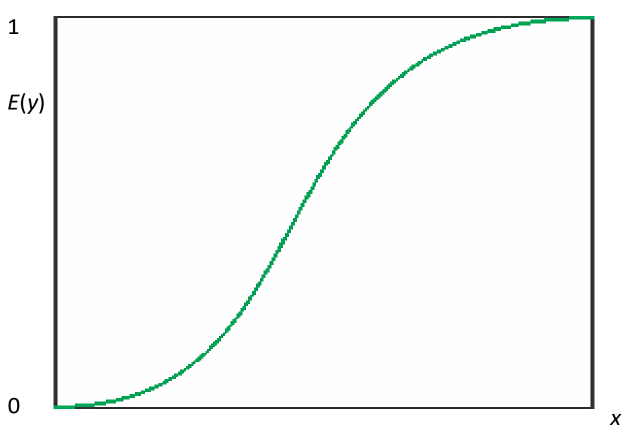
Recall that earlier ε_i was assumed to follow a normal distribution when y was not an indicator variable. When y is an indicator variable, then ε_i takes only two values, so it cannot be assumed to follow a normal distribution.

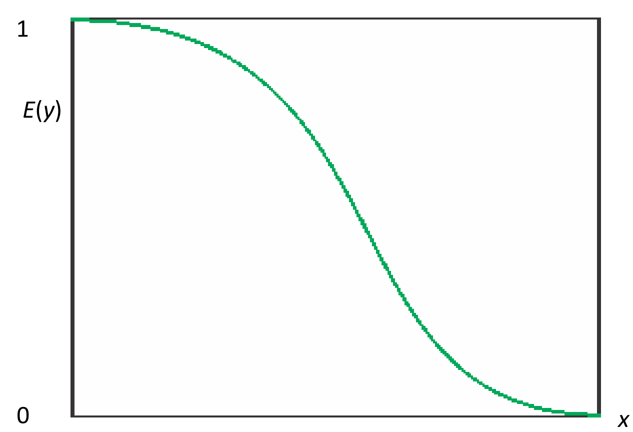
In the usual regression model, the errors are homoskedastic, i.e., $Var(\varepsilon_i) = \sigma^2$ and so $Var(y_i) = \sigma^2$. When y is an indicator variable, then

$$\begin{aligned} Var(y_i) &= E\left[y_i - E(y_i)\right]^2 \\ &= (1-\pi_i)^2\pi_i + (0-\pi_i)^2(1-\pi_i) \\ &= \pi_i(1-\pi_i)\left[1-\pi_i+\pi_i\right] \\ &= \pi_i(1-\pi_i) \\ &= E(y_i)\left[1-E(y_i)\right] \\ &= \sigma^2_{y_i}\ . \end{aligned}$$

Thus $Var(y_i)$ depends on y_i and is a function mean of y_i. Moreover, since $E(y_i) = \pi_i$ and π_i is the probability, so $0 \le \pi_i \le 1$ and thus there is a constraint on $E(y_i)$ that $0 \le E(y_i) \le 1$. This puts a big constraint on the choice of the linear response function. One cannot fit a model in which the predicted values lie outside the interval of 0 and 1.

When y is a dichotomous variable, then empirical pieces of evidence suggest that the function $E(y)$ on the whole real line that can be mapped to $[0,1]$ has the sigmoid shape. It is a nonlinear $S-$shape like





A natural choice for $E(y)$ would be the cumulative distribution function of a random variable. In particular, the logistic distribution, whose cumulative distribution function is the simplified logistic function, yields a good link and is given by

$$\begin{aligned} E(y) &= \frac{\exp(y)}{1+\exp(y)} \\ &= \frac{\exp(x'\beta)}{1+\exp(x'\beta)} \\ &= \frac{1}{1+\exp(-x'\beta)}. \end{aligned}$$

Linear predictor and link functions:

The systematic component in $E(y)$ is the linear predictor and is denoted as

$$\eta_i = \sum_j \beta_j x_{ij} = x_i^{'}\beta, \ i = 1, 2, ..., n, \ j = 0, 1, 2, ..., k.$$

The link function in generalized linear model relates the linear predictor η_i to the mean response μ_i.

Thus

$$\begin{aligned} &g(\mu_i) = \eta_i \\ &\text{or } \mu_i = g^{-1}(\eta_i). \end{aligned}$$

In the usual linear models based on the normally distributed study variable, the link $g(\mu_i) = \mu_i$ is used and is called an **identity link**. A link function maps the range of μ_i onto the whole real line, provides a good empirical approximation and carries meaningful interpretations in real applications.

In the case of logistic regression, the link function is defined as

$$\eta = \ln \frac{\pi}{1-\pi}.$$

This transformation is called the **logit** transformation of probability π and $\frac{\pi}{1-\pi}$ is called as **odds**. The link η is also called as **log-odds**. This link function is obtained as follows:

$$\pi = \frac{1}{1+\exp(-\eta)}$$

or $\pi\left[1+\exp(-\eta)\right]=1$

or $e^{-\eta} = \frac{1-\pi}{\pi}$

or $\mu = \ln\frac{\pi}{1-\pi}$.

Note: Similar to the logit function, there are other functions with the same shape as the logistic function. These functions can also be transformed through π. There are two such popular transformations: the probit and complementary log-log. The probit transformation is based on the transformation of π using the cumulative distribution function of the normal distribution, and based on this is the **probit regression model.**

The **complementary log-log transformation of** π is $\ln\left[-\ln(1-\pi)\right]$.

Maximum likelihood estimation of parameters:

Consider the general form of the logistic regression model

$$y_i = E(y_i) + \varepsilon_i$$

where y_i's are independent Bernoulli random variables with a parameter π_i with

$$E(y_i) = \pi_i$$
$$= \frac{\exp(x_i^{'}\beta)}{1+\exp(x_i^{i}\beta)}$$

The probability density function of y_i is

$$f_i(y_i) = \pi_i^{y_i}(1-\pi_i)^{1-y_i}, i = 1,2,...,n,\ y_i = 0 \text{ or } 1.$$

The likelihood function is

$$L(y_1, y_2,..., y_n, \beta_1, \beta_2,..., \beta_k) = L = \prod_{i=1}^{n} f_i(y_i)$$
$$= \prod_{i=1}^{n} f_i(y_i)(1-\pi_i)^{1-y_i}$$

$$\ln L = \sum_{i=1}^{n} \left[\ln \pi_i^{y_i} + \ln(1-\pi_i)^{1-y_i} \right]$$
$$= \sum_{i=1}^{n} \left[y_i \ln \pi_i + (1-y_i) \ln(1-\pi_i) \right]$$
$$= \sum_{i=1}^{n} \left[y_i \ln \left(\frac{\pi_i}{1-\pi_i} \right) \right] + \sum_{i=1}^{n} \left[\ln(1-\pi_i) \right].$$

Since

$$\pi_i = \frac{\exp(x_i'\beta)}{1+\exp(x_i'\beta)},$$
$$1-\pi_i = \frac{1}{1+\exp(x_i'\beta)},$$
$$\frac{\pi_i}{1-\pi_i} = \exp(x_i'\beta),$$
$$\ln \frac{\pi_i}{1-\pi_i} = \exp x_i'\beta \quad ,$$

so

$$\ln L = \sum_{i=1}^{n} y_i x_i'\beta - \sum_{i=1}^{n} \ln\left[1+\exp(x_i'\beta)\right].$$

Suppose repeated observations are available at each level of the x-variables. Let y_i be the number of 1's observed for i^{th} observation and n_i be the number of trials at each observation. Then

$$\ln L = \sum_{i=1}^{n} y_i \pi_i + \sum_{i=1}^{n} n_i \ln(1-\pi_i) - \sum_{i=1}^{n} y_i \ln(1-\pi_i).$$

The maximum likelihood estimate $\hat{\beta}$ of β is obtained by the numerical maximization.

If $V(\varepsilon) = \Omega,$ then asymptotically

$$E(\hat{\beta}) = \beta$$
$$V(\hat{\beta}) = (X'\Omega^{-1}X)^{-1}.$$

After obtaining $\hat{\beta}$, the linear predictor is estimated by

$$\hat{\eta}_i = x_i'\beta.$$

The fitted value is

$$\hat{y}_i = \hat{\pi}_i = \frac{\exp(\hat{\eta}_i)}{1+\exp(\hat{\eta}_i)} = \frac{1}{1+\exp(-\hat{\eta}_i)} = \frac{1}{1+\exp(-x_i'\hat{\beta})}.$$

Interpretation of parameters:

To understand the interpretation of the related $\beta's$ in the logistic regression model, first, consider a simple case with only one variable as

$$\eta(x) = \beta_0 + \beta_1 x.$$

After fitting the model, $\hat{\beta}_0$ and $\hat{\beta}_1$ are obtained as the estimators of β_0 and β_1 respectively. Then the fitted linear predictor at $x = x_i$ is

$$\hat{\eta}(x_i) = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

which is the log-odds at $x = x_i$. The fitted value at $x = x_i + 1$ is

$$\hat{\eta}(x_i + 1) = \hat{\beta}_0 + \hat{\beta}_1 (x_i + 1)$$

which is the log-odds at $x = x_i + 1$.

Thus

$$\begin{aligned} \hat{\beta}_1 &= \hat{\eta}(x_i + 1) - \hat{\eta}(x_i) \\ &= \ln\left[\text{odds}(x_i + 1)\right] - \ln\left[\text{odds}(x_i)\right] \\ &= \ln\left[\frac{\text{odds}(x_i + 1)}{\text{odds}(x_i)}\right] \\ \Rightarrow \frac{\text{odds}(x_i + 1)}{\text{odds}(x_i)} &= \exp(\hat{\beta}_1). \end{aligned}$$

This is termed as **the odds ratio, which is the estimated increase in the probability of success when the value of the** explanatory variable changes by one unit.

When there are more than one explanatory variables in the model, then the interpretation of $\beta_j's$ is similar to the case of a single explanatory variable case. The odds ratio is $\exp(\hat{\beta}_j)$ associated with the explanatory variable x_j keeping other explanatory variables constant. This is similar to the interpretation of β_j in a multiple linear regression model.

If there is a m unit change is the explanatory variable, then the estimated increase in odds ratio is $\exp(m\hat{\beta}_j)$.

Test of hypothesis:

The test of the hypothesis for the parameters in the logistic regression model is based on asymptotic theory. It is a large-sample test based on the likelihood ratio test, using a statistic called **deviance.**

A model with exactly p parameters that perfectly fit the sample data is termed as a **saturated model.**

The statistic that compares the log-likelihoods of fitted and saturated models is called as **model deviance**. It is defined as

$$\lambda(\beta) = 2\ln L(\text{saturated model}) - 2\ln L(\hat{\beta})$$

where $\ln L(\cdot)$ is the log-likelihood and $\hat{\beta}$ is the maximum likelihood estimate of β.

In the case of the logistic regression model, $y_i = 0$ or 1 and π_i's are completely unrestricted. So the likelihood will be maximum at $\pi_i = y_i$, and the maximum value of L (saturated modal) is

$$Maximum\ L(\text{saturated model}) = 1$$
$$\Rightarrow \ln Maximum\, L(\text{saturated model}) = 0.$$

Let $\hat{\beta}$ be the maximum likelihood estimator of β, then the log-likelihood is maximum at $\beta = \hat{\beta}$, and

$$\begin{aligned}\ln L(\hat{\beta}) &= \sum_{i=1}^{n} y_i x_i' \hat{\beta}_i - \sum_{i=1}^{n} \ln\left[1 + \exp(x_i'\beta)\right] \\ &\geq \ln L(\text{saturated model}).\end{aligned}$$

Assuming that the logistic regression function is correct, the large sample distribution of the likelihood ratio test statistic $\lambda(\beta)$ is approximately distributed as $\chi^2(n-p)$, when n is large.

A large value of $\lambda(\beta)$ implies the model is incorrect. A small value of $\lambda(\beta)$ implies that the model is well-fitted and as good as the saturated model. Note that, in general, the fitted model will have fewer parameters than the saturated model, which includes all parameters. Thus at $\alpha\%$ level of significance.

Poisson Regression Models:

The usual regression model assumes that the random errors are normally distributed, and hence that the study variable is normally distributed. If the study variable is dichotomous (0, 1), logistic regression is used, assuming it follows a Bernoulli distribution.

Similarly, we consider situations in which the study variable is a count representing the number of a relatively rare event. For example, the study variable can be a count of patients with a rare disease, with one or more explanatory variables such as age, haemoglobin level, blood sugar, etc. In another example, the study variable can be the number of defects in the car engine of a reputed car maker, which again depends on one or more explanatory variables.

Assumption of normal or Bernoulli distribution for the study variable will not be appropriate in such situations. The Poisson distribution better describes such situations. So we assume that the study variable y_i is a count variable and follows a Poisson distribution with parameter $\lambda > 0$ as

$$p(y) = \frac{e^{-\lambda}\lambda^{y}}{y!}, \quad y = 0, 1, 2, ...$$

Note that the mean and variance of a Poisson random variable are the same and related as

$$E(y) = \lambda, \; Var(y) = \lambda.$$

Based on a sample $y_1, y_2, ..., y_n$, we can write

$$E(y_i) = \lambda$$

and express the Poisson regression model as

$$y_i = E(y_i) + \varepsilon_i, \; i = 1, 2, ..., n$$

where ε_i's are disturbance terms.

We can define a link function g that relates to the mean of the study variable to a linear predictor as

$$\begin{aligned} g(\lambda_i) &= \eta_i \\ &= \beta_0 + \beta_1 x_1 + ... + \beta_k x_k \\ &= x_i'\beta \end{aligned}$$

and

$$\begin{aligned} \lambda_i &= g^{-1}(\eta_i) \\ &= g^{-1}(x_i'\beta). \end{aligned}$$

The **identity link function** is

$$g(\lambda_i) = \lambda_i = x_i^{'}\beta.$$

The **log-link function** is

$$g(\lambda_i) = \ln(\lambda_i) = x_i^{'}\beta$$

$$\Rightarrow \lambda_i = g^{-1}(x_i^{'}\beta) = \exp(x_i^{'}\beta).$$

Note that in the identity link function, the predicted values of y can be negative, but in the log-link function, the predicted values of y are nonnegative.

Maximum likelihood estimation of parameters:

We use maximum likelihood estimation to estimate the parameters of the Poisson regression model. The likelihood function is based on the Poisson distribution with parameter λ and then $\beta's$ are estimated through the link function.

The likelihood function of $y_1, y_2, ..., y_n$ is

$$\begin{aligned} L(y,\lambda) &= \prod_{i=1}^{n} p_i(y_i) \\ &= \prod_{i=1}^{n} \frac{\exp(-\lambda_i)\lambda_i^{y_i}}{y_i!} \\ &= \frac{\left(\prod_{i=1}^{n} \lambda_i^{\mu_i}\right)\left(\exp\left(-\sum_{i=1}^{n}\lambda_i\right)\right)}{\prod_{i=1}^{n} y_i!} \end{aligned}$$

$$\ln\ L(y,\lambda) = \sum_{i=1}^{n} y_i \ln(\lambda_i) - \sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n}\ln(y_i!).$$

The parameter λ_i can be related to $\beta's$ through the link function

$$\lambda_i = g^{-1}(x_i^{'}\beta).$$

After choosing the proper link function, the log-likelihood function can be maximized using some numerical optimization techniques for a given set of data. Let $\hat{\beta}$ be the obtained maximum likelihood estimator of β. Then the fitted Poisson regression model is

$$\hat{y}_i = g^{-1}(x_i^{'}\hat{\beta}).$$

In case of **identity link,**

$$\hat{y}_i = g^{-1}(x_i'\beta) = x_i'\beta.$$

In the case of **log-link,**

$$\hat{y}_i = g^{-1}(x_i'\hat{\beta}) = \exp(x_i'\hat{\beta}) .$$

Testing of hypothesis:

The test of the hypothesis for the Poisson regression model is similar to that for the logistic regression model. It is constructed as **a model deviance,** which is based on a large sample test using the likelihood ratio test statistic.

The model deviance is defined as

$$\lambda^*(\beta) = 2\ln L(\text{saturated model}) - 2\ln L(\hat{\beta})$$

where the saturated model is based on all the p parameters of the model, and it fits the data perfectly.

The statistic $\lambda^*(\beta)$ has approximately $\chi^2(n-p)$ distribution when n is large. The large value of $\lambda^*(\beta)$ indicates that the model is not correctly fitted to the given data, whereas small values of $\lambda^*(\beta)$ indicating that the model is well fitted to the given set of data in the sense that it is as good as the saturated model.

If $\lambda(\beta) \le \chi^2_{n-p}(\alpha) \Rightarrow$ the fitted model is adequate

and if $\lambda(\beta) > \chi^2_{n-p}(\alpha) \Rightarrow$ the fitted model is not adequate

at $\alpha\%$ level of significance.