Ridge Regression Estimation Approach to Measurement Error Model

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Abstract

This paper considers the estimation of the parameters of measurement error models where the estimated covariance matrix of the regression parameters is ill conditioned. We consider the Hoerl and Kennard type (1970) ridge regression (RR) modifications of the five quasi-empirical Bayes estimators of the regression parameters of a measurement error model when it is suspected that the parameters may belong to a linear subspace. The modifications are based on the estimated covariance matrix of the estimators of regression parameters. The estimators are compared and the dominance conditions as well as the regions of optimality of the proposed estimators are determined based on quadratic risks.

Keywords: Linear Regression model, measurement error, multicollinearity, reliability matrix, ridge regression estimators, shrinkage estimation, Stein type estimators, preliminary test estimator, asymptotic relative efficiency.

1 Introduction

One of the basic assumptions in the linear regression analysis is that all the explanatory variables are linearly independent, i.e., the rank of the matrix of the observations on all the explanatory variable is of full column rank. When this assumption is violated, the problem of multicollinearity enters into the data and it inflates the variance of ordinary least squares estimator of the regression coefficient which is the minimum in the absence of multicollinearity. The problem of multicollinearity has attracted many researchers, see see Silvey (1969) and large amount of research has been devoted to diagnose the presence and degree of multicollinearity, see Belsley (1991), Belsley, Kuh and Welsh (2004) and Rao, Toutenburg, Shalabh and Heumann (2008) for more details. A complete bibliography on multicollinearity is not one of the objectives of this paper. Obtaining the estimators for multicollinear data is an important problem in the literature. Several approaches have been proposed for its solution. Among them, the ridge regression estimation approach due to Hoerl and Kennard (1970) turned out to be the most popular approach among researchers as well as practitioners. The ridge estimators under the normally distributed random errors in regression model have been studied by Gibbons (1981), Sarker (1992), Saleh and Kibria (1993), Gruber (1998), Malthouse (1999), Singh and Tracy (1999), Wencheko (2000), Inoue (2001), Kibria and Saleh (2003, 2004), Arashi, Tabatabaey and Iranmanesh (2010), Hassanzadeh, Arashi and Tabatabaey (2011a, 2011b), Arashi, Tabatabaey and Soleimani (2012), Bashtian, Arashi and Tabatabaey (2011a, 2011b) etc. The details of development of other approaches and the literature related to the ridge regressions is not within the scope of this paper.

Another fundamental assumption in all statistical analysis is that all the observations are correctly observed. This assumption is more oftenly violated in real life data. Consequently, the measurement errors creep into the data. In such situations, the observations on the variables are not correctly observable and are contaminated with measurement errors. The usual statistical tools tend to loose their validity when the data is contaminated with measurement errors, see Fuller (1987), Cheng and van Ness (1999) for more details on measurement error models. In the context of linear regression models, the ordinary least squares estimator of regression coefficient becomes biased as well as inconsistent in the presence of measurement errors in the data. An important issue in the area of measurement errors is to find the consistent estimators of the parameters. The consistent estimators of the regression coefficient can be obtained by utilizing some additional information from outside the sample. In the context of multiple linear regression models, the additional information in the form of known covariance matrix of measurement errors associated with explanatory variables and known matrix of reliability ratios of explanatory variables have been extensively utilized in various context, see e.g., Gleser(1992, 1993), Kim and Saleh (2002, 2003a, 2003b, 2005, 2008, 2011), Saleh (2010), Sen and Saleh (2010), Jurečkov*á*, Sen and Saleh (2010), Shalabh, Garg, Misra (2009) etc.

In many practical situations, the observations are not only measurement error ridden but also exhibit some linear relationships among them. So the problem of multicollinearity enter into the measurement error ridden data. An important issue crops up then is how to obtain the estimators of regression coefficients under such a situation. One simple idea is to use the ridge regression estimation over the measurement error ridden data. An obvious question that crops up is what happens then? In this paper, we attempt to answer such questions.

It is well known that Stein (1956) and James and Stein (1961) initially proposed the Stein estimator and positive-rule estimators which have been justified by empirical Bayes methodology due to Efron (1975, 1977), Efron and Morris (1972, 1973a, 1973b, 1975). The preliminary test estimators were proposed by Bancroft (1944). On the other hand, ridge regression estimators were proposed by Hoerl and Kennard (1973) and they combat the problem of multicollinearity for the estimation of regression parameters. Saleh (2006, Chapter 4) proposed "quasi-empirical Bayes estimators". So we have considered five quasi-empirical Bayes estimators estimators by weighing the unrestricted, restricted, preliminary test and Stein-type estimators by the ridge "weight function". The resulting estimators are studied in measurement error models. The quadratic risks of these estimators have been obtained and optimal regions of superiority of the estimators are determined.

The plan of the paper is as follows. We describe the model set up in Section 2. The details and development of the estimators are presented in Section 3. The comparison of estimators over each other is studied and their dominance conditions are reported in Section 4. The summary and conclusions are placed in Section 5 followed by the references.

2 The Model Description

Consider the multiple regression model with measurement errors

$$\left.\begin{array}{lcl} Y_t & = & \beta_0 + \boldsymbol{x}_t'\boldsymbol{\beta} + e_t, \\ \boldsymbol{X}_t & = & \boldsymbol{x}_t + \boldsymbol{u}_t \end{array}\right\} t = 1, 2, \ldots, n \tag{2.1}$$

where β_0 is the intercept term and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)'$ is the $p \times 1$ vector of regression coefficients, $\boldsymbol{x}_t = (x_{1t}, x_{2t}, \ldots, x_{pt})'$ is the $p \times 1$ vector of set t^{th} observations on true but unobservable p explanatory variables that are observed as $\boldsymbol{X}_t = (X_{1t}, X_{2t}, \ldots, X_{pt})'$ with $p \times 1$ measurement error vector $\boldsymbol{u}_t = (u_{1t}, u_{2t}, \ldots, u_{pt})'$, u_{it} being the measurement error in i^{th} explanatory variable x_{it} and e_t is the response error in observed response variable Y_t. We assume that

$$(\boldsymbol{x}_t', e_t, \boldsymbol{u}_t) \sim N_{2p+1}\left\{(\boldsymbol{\mu}_x', 0, \boldsymbol{0}')', \mathrm{BlockDiag}(\Sigma_{xx}, \sigma_{ee}, \Sigma_{uu})\right\} \tag{2.2}$$

with $\boldsymbol{\mu}_x = (\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_p})'$, σ_{ee} is the variance of e_t's whereas Σ_{xx} and Σ_{uu} are the covariance matrices of x_t's and u_t's respectively. Clearly, $(Y_t, \boldsymbol{X}_t')'$ follows a $(p+1)$-variate normal distribution with mean vector $(\beta_0 + \boldsymbol{\beta}'\boldsymbol{\mu_x}, \boldsymbol{\mu_x'})'$ and covariance matrix

$$\begin{pmatrix} \sigma_{ee} + \boldsymbol{\beta}'\Sigma_{xx}\boldsymbol{\beta} & \boldsymbol{\beta}'\Sigma_{xx} \\ \Sigma_{xx}\boldsymbol{\beta} & \Sigma_{xx} + \Sigma_{uu} \end{pmatrix}. \tag{2.3}$$

Then the conditional distribution of Y_t given $\boldsymbol{X}_t$ is

$$N\left\{\gamma_0 + \boldsymbol{\gamma}'\boldsymbol{X}_t; \sigma_{zz} = \sigma_{ee} + \boldsymbol{\beta}'\Sigma_{xx}(I_p - K_{xx}')\beta\right\} \tag{2.4}$$

where

$$K_{xx} = \Sigma_{XX}^{-1}\Sigma_{xx} = (\Sigma_{xx} + \Sigma_{uu})^{-1}\Sigma_{xx} \tag{2.5}$$

is the $p \times p$ matrix of reliability ratios of X, see Gleser(1992).

Further, we note that

$$\left.\begin{array}{rcl}\gamma_0 & = & \beta_0 + \boldsymbol{\beta}'(I_p - K'_{xx})\boldsymbol{\mu}_x, \\ \boldsymbol{\gamma} & = & K_{xx}\boldsymbol{\beta} \text{ and} \\ \boldsymbol{\beta} & = & K_{xx}^{-1}\boldsymbol{\gamma}.\end{array}\right\} \tag{2.6}$$

Our basic problem is the estimation of $\boldsymbol{\beta}$ under various situations situations beginning with the primary estimation of $\boldsymbol{\beta}$ assuming Σ_{uu} is known which is as follows.

Let

$$S = \begin{pmatrix} S_{YY} & S_{Y\boldsymbol{x}} \\ S_{xY} & S_{XX} \end{pmatrix} \tag{2.7}$$

where

(i) $S_{YY} = (\boldsymbol{Y} - \bar{Y}\mathbf{1}_p)'(\boldsymbol{Y} - \bar{Y}\mathbf{1}_p), Y = (Y_1, Y_2, \ldots, Y_n)', \mathbf{1}_n = (1, 1, \ldots, 1)'$.

(ii) $S_{XX} = ((S_{X_iX_i})), S_{X_iX_i} = (\boldsymbol{x}_i - \bar{X}_i\mathbf{1}_n)'(\boldsymbol{x}_i - \bar{X}_i\mathbf{1}_n)$

(iii) $S_{\boldsymbol{X}_iY} = (\boldsymbol{X}_i - \bar{X}_i\mathbf{1}_n)'(\boldsymbol{Y}_i - \bar{Y}_i\mathbf{1}_n), S_{XY} = (S_{X_1Y}, S_{X_2Y}, \ldots, S_{X_pY})'$

(iv) $\bar{X}_i = \frac{1}{n}\sum_{t=1}^n X_{it}, \bar{Y} = \frac{1}{n}\sum_{t=1}^n Y_t$.

Clearly, $\frac{1}{n-1}S_{XX}$ is an unbiased estimator of Σ_{XX} and $\frac{1}{n}S_{XX} \xrightarrow{P} \Sigma_{XX}$ as $n \to \infty$ where $\xrightarrow{P}$ denotes the convergence in probability. Gleser (1992) showed that the maximum likelihood estimators of γ_0, $\boldsymbol{\gamma}$ and σ_{zz} are just the naive least squares estimators, viz.,

$$\left.\begin{array}{rcl}\tilde{\gamma}_{0n} & = & \bar{Y} - \tilde{\boldsymbol{\gamma}}'_n\bar{\boldsymbol{X}}, \\ \tilde{\boldsymbol{\gamma}}_n & = & S_{XX}^{-1}S_{XY} \quad \text{and} \\ \tilde{\sigma}_{zz} & = & \frac{1}{n}(Y - \tilde{\gamma}_{0n}\mathbf{1}_n - \tilde{\boldsymbol{\gamma}}'_n\boldsymbol{X})'(Y - \tilde{\gamma}_{0n}\mathbf{1}_n - \tilde{\boldsymbol{\gamma}}'_n\boldsymbol{X})\end{array}\right\} \tag{2.8}$$

provided

$$\tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\boldsymbol{\gamma}}'_n K_{xx}^{-1}\Sigma_{uu}\tilde{\boldsymbol{\gamma}}_n \geq 0. \tag{2.9}$$

When Σ_{uu} is known and K_{xx} is unknown, then K_{xx} is estimated by

$$\hat{K}_{xx} = S_{XX}^{-1}(S_{XX} - n\Sigma_{uu}). \tag{2.10}$$

Further, $\hat{K}_{xx} \xrightarrow{P} K_{xx}$ as $n \to \infty$. Thus, the maximum likelihood estimates of β_0, $\boldsymbol{\beta}_1$

and σ_{ee} are given by

$$\left.\begin{array}{rcl}\tilde{\beta}_{0n} & = & \tilde{\gamma}_{on} - \tilde{\boldsymbol{\beta}}_n'(I_p - \hat{K}_{xx}')\bar{\boldsymbol{x}}, \\ \tilde{\boldsymbol{\beta}}_n & = & \hat{K}_{xx}^{-1}\tilde{\boldsymbol{\gamma}}_n \quad \text{and} \\ \tilde{\sigma}_{ee} & = & \tilde{\sigma}_{zz} - \tilde{\boldsymbol{\beta}}_n'\Sigma_{uu}\hat{K}_{xx}\tilde{\beta}_n\end{array}\right\} \tag{2.11}$$

respectively.

Finally, $\tilde{\beta}_{0n}$ reduces to $\bar{Y} - \tilde{\boldsymbol{\beta}}_n'\bar{\boldsymbol{X}}$ and

$$\tilde{\boldsymbol{\beta}}_n = (S_{XX} - n\Sigma_{uu})^{-1}S_{XY} \tag{2.12}$$

provided $\tilde{\sigma}_{ee} \geq 0$ as in (2.9). The estimators will be designated as the unrestricted estimators of β_0 and $\boldsymbol{\beta}$. Then by Theorem 2 of Fuller (1987) we have as $n \to \infty$

(i) $\sqrt{n}(\tilde{\boldsymbol{\gamma}}_n - \boldsymbol{\gamma})$ is normally distributed with mean 0 and covariance matrix $\sigma_{zz}\Sigma_{XX}^{-1}$.

(ii) $\sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta})$ is normally distributed with mean $\mathbf{0}$ and covariance matrix $\sigma_{zz}C^{-1}$ defined by

$$C = K_{xx}'\Sigma_{XX}K_{xx} \tag{2.13}$$

Let C_n be a consistent estimator of C given by

$$C_n = \hat{K}_{xx}'\hat{\Sigma}_{XX}\hat{K}_{xx} = (S_{XX} - n\Sigma_{uu})'\, S_{XX}^{-1}\,(S_{XX} - n\Sigma_{uu})$$

Thus, using (2.10) we may write

$$\frac{1}{n}C_n = C + O_p(1).$$

In case, $\boldsymbol{\beta}$ is suspected to belong to the linear subspace of $H\boldsymbol{\beta} = \boldsymbol{h}$ where H is a $q \times p$ matrix and $\boldsymbol{h}$ is a $q \times 1$ vector of known numbers respectively, the restricted estimator of $\boldsymbol{\beta}$ is defined by

$$\hat{\boldsymbol{\beta}}_n = \tilde{\boldsymbol{\beta}}_n - C_n^{-1}H'(HC_n^{-1}H')^{-1}(H\tilde{\boldsymbol{\beta}}_n - \boldsymbol{h}), \tag{2.14}$$

see Saleh and Shiraish (1989).

Under H_0, as $n \to \infty$, we have

$$\begin{pmatrix}\sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \\ \sqrt{n}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \\ \sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n)\end{pmatrix} \xrightarrow{D} N_{3p}\left\{\begin{pmatrix}\mathbf{0} \\ \mathbf{0} \\ \mathbf{0}\end{pmatrix}, \sigma_{zz}\begin{pmatrix}C^{-1} & C^{-1} - A & A \\ C^{-1} - A & C^{-1} - A & \mathbf{0} \\ A & \mathbf{0} & A\end{pmatrix}\right\} \tag{2.15}$$

where $A = C^{-1}H'(HC^{-1}H')^{-1}HC^{-1}$.

Since it is suspected that the restrictions $H\boldsymbol{\beta} = \boldsymbol{h}$ may hold, we remove the suspicion by testing hypothesis H_0 based on the Wald-type statistic

$$\mathcal{L}_n^* = \frac{(H\tilde{\boldsymbol{\beta}}_n - h)'(HC_n^{-1}H')^{-1}(H\tilde{\boldsymbol{\beta}}_n - h)}{\tilde{\sigma}_{zz}}. \tag{2.16}$$

Thus under H_0, as $n \to \infty$, $\mathcal{L}_n^* \xrightarrow{D} \chi_q^2$, chi-square variable with q degrees of freedom where $\xrightarrow{D}$ denotes the convergence in distribution.

3 Regression Estimators of $\boldsymbol{\beta}$

In this section, we introduce the ridge regression estimators of $\boldsymbol{\beta}$. For this, we first consider the conditional set up of least squares method with known reliability matrix K_{xx} and minimize the quadratic form with Lagrangian multiplier

$$(XK\boldsymbol{\beta} + \gamma_0\mathbf{1}_p - Y)'(XK\boldsymbol{\beta} + \gamma_0\mathbf{1}_p - Y) + k\boldsymbol{\beta}'\boldsymbol{\beta}.$$

This minimization yields the normal equation for $\boldsymbol{\beta}$ as

$$[K_{xx}'S_{XX}K_{xx} + kI_p]\,\boldsymbol{\beta} = K_{xx}'X'Y.$$

Thus the ridge regression estimator for $\boldsymbol{\beta}$ is given by

$$\tilde{\boldsymbol{\beta}}_n(k) = \left[I_p + \left(\hat{K}_{xx}'S_{XX}\hat{K}_{xx}\right)^{-1}\right]^{-1}\hat{\boldsymbol{\beta}}_n, \tag{3.1}$$

substituting the estimators of K_{xx} given by (2.10) with

$$\tilde{\boldsymbol{\beta}}_n = (S_{xx} - n\Sigma_{uu})^{-1}\,S_{xy}.$$

Here, the ridge factor of the ridge estimator is given by

$$R_n(k) = \left[I_p + kC_n^{-1}\right]^{-1}, \quad C_n = \hat{K}_{xx}'S_{xx}\hat{K}_{xx} \tag{3.2}$$

which is a consistent estimator of

$$R(k) = \left[I_p + kC^{-1}\right]^{-1}, \quad C = K_{xx}'\Sigma_{XX}K_{xx}.$$

Based on this factor, the unrestricted ridge regression estimator $\tilde{\boldsymbol{\beta}}_n(k)$ is defined by

$$\tilde{\boldsymbol{\beta}}_n(k) = R_n(k)\tilde{\boldsymbol{\beta}}_n.$$

It is easy to verify that as $n \to \infty$, the bias, MSE and trace of MSE expressions for $\tilde{\boldsymbol{\beta}}_n(k)$ are given by

$$\begin{aligned}
b_1(\tilde{\boldsymbol{\beta}}_n(k)) &= -kC^{-1}(k)\boldsymbol{\beta}\ ;\ C^{-1}(k) = (C + kI_p)^{-1} \\
M_1(\tilde{\boldsymbol{\beta}}_n(k)) &= \sigma_{zz}[R(k)]'C^{-1}[R(k)] + k^2C^{-1}(k)\boldsymbol{\beta}\boldsymbol{\beta}'C^{-1}(k) \\
tr(M_1(\tilde{\boldsymbol{\beta}}_n(k))) &= \sigma_{zz}tr\{[R(k)]'C^{-1}[R(k)]\} + k^2\boldsymbol{\beta}'C^{-2}(k)\boldsymbol{\beta}.
\end{aligned}$$

Further, since $\boldsymbol{\beta}$ is suspected to belong to the subspace $H\boldsymbol{\beta} = \boldsymbol{h}$, we shall consider four more estimators, viz., the

(i) Restricted estimator of $\boldsymbol{\beta}$ given by

$$\hat{\boldsymbol{\beta}}_n(k) = R_n(k)\hat{\boldsymbol{\beta}}_n \tag{3.3}$$

(ii) Preliminary test estimator (PTE) of $\boldsymbol{\beta}$ given by

$$\hat{\boldsymbol{\beta}}_n^{PT}(k) = R_n(k)\hat{\boldsymbol{\beta}}_n^{PT} \tag{3.4}$$

where

$$\hat{\boldsymbol{\beta}}_n^{PT} = \tilde{\boldsymbol{\beta}}_n - (\tilde{\boldsymbol{\beta}}_n - \hat{\boldsymbol{\beta}}_n)I(\boldsymbol{\mathcal{L}}_n^* < \chi_q^2(\alpha)) \tag{3.5}$$

and $\chi_q^2(\alpha)$ denotes the α-level critical value of a Chi-square distribution with q degrees of freedom.

The preliminary test estimation under the assumption of normally distributed random errors has been pioneered by Bancroft (1944) and considered later by Bancroft (1964), Han and Bancroft (1968), Judge and Bock (1978), Giles (1991), Benda (1996), Kibria and Saleh (2006), Saleh (2006) and Arashi (2009), Arashi, Tabatabaey and Soleimani (2012) among others. In the set up of measurement errors models, Kim and Saleh (2002, 2003a, 2003b, 2005, 2008, 2011) have considered the preliminary test and Stein-type estimation.

(iii) James-Stein type shrinkage estimator (SE) of $\boldsymbol{\beta}$ due to James and Stein (1961) is given by

$$\hat{\boldsymbol{\beta}}_n^{S}(k) = R_n(k)\hat{\boldsymbol{\beta}}_n^{S} \tag{3.6}$$

where

$$\hat{\boldsymbol{\beta}}_n^S = \tilde{\boldsymbol{\beta}}_n - (q-2)(\tilde{\boldsymbol{\beta}}_n - \hat{\boldsymbol{\beta}}_n)\mathcal{L}_n^{*-1}. \tag{3.7}$$

The Stein-rule estimation technique in various models have been considered by several researchers, see e.g., Ohtani (1993), Gruber (1998), Saleh (2006), Shalabh (1998, 2001), Tabatabaey and Arashi (2008), Arashi and Tabatabaey (2009, 2010a, 2010b) and Hassanzadeh, Arashi and Tabatabaey (2011), Arashi (2012) among many others.

(iv) Positive rule Stein estimator(PRSE) of $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\beta}}_n^{S+}(k) = R_n(k)\hat{\boldsymbol{\beta}}_n^{S+} \tag{3.8}$$

where

$$\hat{\boldsymbol{\beta}}_n^{S+} = \hat{\boldsymbol{\beta}}_n I(\mathcal{L}_n^* < q-2) + \hat{\boldsymbol{\beta}}_n^S I(\mathcal{L}_n^* \geq q-2). \tag{3.9}$$

Now, we present the asymptotic distributional properties of the five ridge regression estimators. It may be verified that the test $\mathcal{L}_n^*$ for the test of $H\boldsymbol{\beta} = \boldsymbol{h}$ is consistent as $n \to \infty$. Thus all the quasi-empirical Bayes estimators $\hat{\boldsymbol{\beta}}_n^{PT}$, $\hat{\boldsymbol{\beta}}_n^S$ and $\hat{\boldsymbol{\beta}}_n^{S+}$ are asymptotically equivalent to $\tilde{\boldsymbol{\beta}}_n$ while the asymptotic distribution of $\hat{\boldsymbol{\beta}}_n$ degenerates as $n \to \infty$. To by pass this problem, we consider the asymptotic distribution under the sequence of local alternatives

$$K_{(n)} : H\boldsymbol{\beta} = \boldsymbol{h} + n^{-\frac{1}{2}}\boldsymbol{\xi}; \;\; \boldsymbol{\xi} \in R^q. \tag{3.10}$$

Thus, using Saleh (2006), we obtain the following Theorem concerning the five basic estimators of $\boldsymbol{\beta}$ defined by (2.12), (2.14), (3.5), (3.7) and (3.9).

Theorem 1 *Under $\{K_{(n)}\}$ and the basic assumptions of the measurement error model, the following holds:*

(a)

$$\begin{pmatrix} \sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \\ \sqrt{n}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \\ \sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \hat{\boldsymbol{\beta}}_n) \end{pmatrix} \xrightarrow{D} N_{3p}\left\{ \begin{pmatrix} 0 \\ -\boldsymbol{\delta} \\ \boldsymbol{\delta} \end{pmatrix}, \sigma_{zz} \begin{pmatrix} C^{-1} & C^{-1}-A & A \\ C^{-1}-A & C^{-1}-A & \mathbf{0} \\ A & \mathbf{0} & A \end{pmatrix} \right\}. \tag{3.11}$$

(b)

$$\lim_{n\to\infty} P\{\mathcal{L}_n \leq x \mid K_{(n)}\} = \mathcal{H}_q(x, \Delta^2), \;\; \Delta^2 = \frac{1}{\sigma_{zz}}\boldsymbol{\delta}' C \boldsymbol{\delta}\ , \;\boldsymbol{\delta} = C^{-1}H'(HC^{-1}H')^{-1}\boldsymbol{\xi}$$

where $\mathcal{H}_q(x;\Delta^2)$ *is the c.d.f. of a noncentral Chi-square distribution with* q *degrees of freedom and noncentrality parameter* Δ^2*.*

(c)

$$\lim_{n\to\infty} P\{\sqrt{n}(\hat{\boldsymbol{\beta}}_n^{PT} - \boldsymbol{\beta}) \le \boldsymbol{x} \mid K_{(n)}\} = \mathcal{H}_q(\chi_q^2(\alpha);\Delta^2) G_p((\boldsymbol{x}+\boldsymbol{\delta}); 0; \sigma_{zz}[C^{-1} - A])$$
$$+ \int_{E(\Delta^2)} G_p(\boldsymbol{x} - C^{-1}H'(HC^{-1}H')^{-1}Z; 0; \sigma_{zz}[C^{-1} - A]) dG_p(Z; 0; \sigma_{zz}[HC^{-1}H'])$$

where $G_p(\boldsymbol{x};\boldsymbol{\mu}_x;\Sigma)$ *is the c.d.f. of a p-variate normal distribution with mean* $\boldsymbol{\mu}_x$ *and covariance matrix* Σ *and*

$$E(\Delta^2) = \left\{Z : (Z+\Delta)'(HC^{-1}H')^{-1}(Z+\Delta) \ge \chi_q^2(\alpha)\right\}$$

with $Z \sim N_p(\mathbf{0}, C^{-1})$*.*

(d)

$$\sqrt{n}(\hat{\boldsymbol{\beta}}_n^{S} - \boldsymbol{\beta}) \xrightarrow{D} \left[Z - \frac{(q-2)C^{-1}H'(HC^{-1}H')^{-1}(HZ+\Delta)}{(Z+\Delta)'(HC^{-1}H')^{-1}(Z+\Delta)}\right].$$

(e)

$$\begin{aligned}\sqrt{n}(\hat{\boldsymbol{\beta}}_n^{S+} - \boldsymbol{\beta}) \quad \xrightarrow{D} \quad & \left[Z - \frac{(q-2)C^{-1}H'(HC^{-1}H')^{-1}(HZ+\Delta)}{(Z+\Delta)'(HC^{-1}H')^{-1}(Z+\Delta)}\right] \\ & \times I\left[(HZ+\Delta)'(HC^{-1}H')^{-1}(HZ+\Delta) \ge (q-2)\right] \\ & + C^{-1}H'(HC^{-1}H')^{-1}(HZ+\Delta) \\ & \times I\left[(HZ+\Delta)'(HC^{-1}H')^{-1}(HZ+\Delta) < (q-2)\right].\end{aligned}$$

Here $\xrightarrow{D}$ *denotes the convergence in distribution.*

Then we have the following theorem giving the characteristics of the basic five estimators of $\boldsymbol{\beta}$. The risk functions are computed using a quadratic loss function with a semi-definite matrix Q. Note that $\boldsymbol{b}(\hat{\boldsymbol{\theta}})$, $M(\hat{\boldsymbol{\theta}})$ and $R(\hat{\boldsymbol{\theta}};\cdot)$ denote the bias, MSE matrix and risks, respectively of an estimator $\hat{\boldsymbol{\theta}}$ of a parameter $\boldsymbol{\theta}$.

Theorem 2 *Under* $\{K_{(n)}\}$ *and the assumed regularity conditions as* $n \to \infty$*, the following holds:*

(a)

$$\begin{aligned}
\boldsymbol{b}_1(\tilde{\boldsymbol{\beta}}_n) &= \mathbf{0}, \\
M_1(\tilde{\boldsymbol{\beta}}_n) &= \sigma_{zz} C^{-1}, \\
R_1(\tilde{\boldsymbol{\beta}}_n; Q) &= \sigma_{zz} tr(C^{-1}Q).
\end{aligned}$$

(b)

$$\begin{aligned}
\boldsymbol{b}_2(\hat{\boldsymbol{\beta}}_n) &= -\boldsymbol{\delta}, \\
M_2(\hat{\boldsymbol{\beta}}_n) &= \sigma_{zz}[C^{-1} - A] + \boldsymbol{\delta}\boldsymbol{\delta}', \\
R_2(\hat{\boldsymbol{\beta}}_n; Q) &= \sigma_{zz} tr[(C^{-1} - A)Q] + \boldsymbol{\delta}' Q \boldsymbol{\delta}.
\end{aligned}$$

(c)

$$\begin{aligned}
b_3(\hat{\boldsymbol{\beta}}_n^{PT}) &= -\boldsymbol{\delta}\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2], \\
M_3(\hat{\boldsymbol{\beta}}_n^{PT}) &= \sigma_{zz} C^{-1} - A\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + \boldsymbol{\delta}\boldsymbol{\delta}'\{2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\}, \\
R_3(\hat{\boldsymbol{\beta}}_n^{PT}; Q) &= \sigma_{zz} tr(C^{-1}Q) - \sigma_{zz} tr(AQ)\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + (\boldsymbol{\delta} Q \boldsymbol{\delta}')\{2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \\
&\quad -\mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\}.
\end{aligned}$$

(d)

$$\begin{aligned}
b_4(\hat{\boldsymbol{\beta}}_n^{S}) &= -(q-2)\boldsymbol{\delta} E\left[\chi_{q+2}^{-2}(\Delta^2)\right], \\
M_4(\hat{\boldsymbol{\beta}}_n^{S}) &= \sigma_{zz} C^{-1} - \sigma_{zz}(q-2)A\left\{2E\left[\chi_{q+2}^{-2}(\Delta^2)\right] - (q-2)E\left[\chi_{q+2}^{-4}(\Delta^2)\right]\right\} \\
&\quad +(q-2)\boldsymbol{\delta}\boldsymbol{\delta}'\left\{2E\left[\chi_{q+2}^{-2}(\Delta^2)\right] - 2E\left[\chi_{q+4}^{-4}(\Delta^2)\right] + (q-2)E\left[\chi_{q+4}^{-4}(\Delta^2)\right]\right\}, \\
R_4(\hat{\boldsymbol{\beta}}_n^{S}; Q) &= \sigma_{zz} tr(C^{-1}Q) - \sigma_{zz}(q-2)tr(AQ)\left\{2E\left[\chi_{q+2}^{-2}(\Delta^2)\right] - (q-2)E\left[\chi_{q+2}^{-2}(\Delta^2)\right]\right\} \\
&\quad +(q-2)\boldsymbol{\delta}' Q \boldsymbol{\delta}\left\{2E\left[\chi_{q+2}^{-2}(\Delta^2)\right] - 2E\left[\chi_{q+4}^{-2}(\Delta^2)\right] + (q-2)E\left[\chi_{q+4}^{-4}(\Delta^2)\right]\right\}.
\end{aligned}$$

(e)

$$
\begin{aligned}
b_5(\hat{\boldsymbol{\beta}}_n^{S+}) &= -(q-2)\boldsymbol{\delta}\left\{E\left[\chi_{q+2}^{-2}(\Delta^2)\right] + \mathcal{H}_{q+2}(q-2;\Delta^2)\right.\\
&\quad \left. -E[\chi_{q+2}^{-2}(\Delta^2)I\left(\chi_{q+2}^2(\Delta^2) < (q-2)\right)]\right\},\\
M_5(\hat{\boldsymbol{\beta}}_n^{S+}) &= M_4(\hat{\boldsymbol{\beta}}_n^{S}) - A\left[\left(1-(q-2)\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < (q-2)\right)\right]\\
&\quad +\boldsymbol{\delta}\boldsymbol{\delta}'\left\{2E\left[\left(1-(q-2)\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < (q-2)\right)\right]\right.\\
&\quad \left. -E\left[\left(1-(q-2)\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^2(\Delta^2) < (q-2)\right)\right]\right\},\\
R_5(\hat{\boldsymbol{\beta}}_n^{S+};Q) &= tr\left[M_4(\hat{\boldsymbol{\beta}}_n^{S})Q\right] - \sigma_{zz}tr(AQ)\left[\left(1-(q-2)\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < (q-2)\right)\right]\\
&\quad +(\boldsymbol{\delta}'Q\boldsymbol{\delta})\left\{2E\left[\left(1-(q-2)\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < (q-2)\right)\right]\right.\\
&\quad \left. -E\left[\left(1-(q-2)\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^2(\Delta^2) < (q-2)\right)\right]\right\}.
\end{aligned}
$$

The following theorem gives the asymptotic distributional bias, MSE matrices and risk expressions for the ridge regression estimators given by (3.3), (3.3), (3.4), (3.6) and (3.8)

Theorem 3 *Under $\{K_{(n)}\}$ and the basic assumptions of the measurement error model, the following holds:*

(a)

$$
\begin{pmatrix} \sqrt{n}(\tilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}) \\ \sqrt{n}(\hat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}) \\ \sqrt{n}(\tilde{\boldsymbol{\beta}}_n(k)-\hat{\boldsymbol{\beta}}_n(k)) \end{pmatrix} \xrightarrow{D} N_{3p}\left\{\begin{pmatrix} -kC^{-1}(k)\boldsymbol{\beta} \\ -kC^{-1}(k)\boldsymbol{\beta} - R(k)\boldsymbol{\delta} \\ R(k)\boldsymbol{\delta} \end{pmatrix}, \sigma_{zz}\Sigma^*\right\} \tag{3.12}
$$

where $C^{-1}(k) = (C+kI_p)^{-1}$ and

$$
\Sigma^* = \begin{pmatrix} \{R(k)C^{-1}R(k)'\} & \{R(k)(C^{-1}-A)R(k)'\} & \{R(k)AR(k)'\} \\ \{R(k)(C^{-1}-A)R(k)'\} & \{R(k)(C^{-1}-A)R(k)'\} & \mathbf{0} \\ \{R(k)AR(k)'\} & \mathbf{0} & \{R(k)AR(k)'\} \end{pmatrix}
$$

(b) The asymptotic distributional bias, MSE matrices and risk expressions are given by

(i)

$$
\begin{aligned}
\boldsymbol{b}_1(\tilde{\boldsymbol{\beta}}_n(k)) &= -kC^{-1}(k)\boldsymbol{\beta},\\
M_1(\tilde{\boldsymbol{\beta}}_n(k)) &= \sigma_{zz}R(k)C^{-1}R(k)' + k^2C^{-1}(k)\boldsymbol{\beta}\boldsymbol{\beta}'C^{-1}(k),\\
R_1(\tilde{\boldsymbol{\beta}}_n(k);W) &= \sigma_{zz}tr\left[WR(k)C^{-1}R(k)'\right] + k^2\boldsymbol{\beta}'C^{-2}(k)\boldsymbol{\beta}.
\end{aligned}
$$

(ii)

$$\begin{aligned}
\boldsymbol{b}_2(\hat{\boldsymbol{\beta}}_n(k)) &= -kC^{-1}(k)\boldsymbol{\beta} - R(k)\boldsymbol{\delta}, \\
M_2(\hat{\boldsymbol{\beta}}_n(k)) &= \sigma_{zz}R(k)[C^{-1} - A]R'(k) + \left[kC^{-1}(k)\boldsymbol{\beta} + R(k)\boldsymbol{\delta}\right]\left[kC^{-1}(k)\boldsymbol{\beta} + R(k)\boldsymbol{\delta}\right]', \\
R_2(\hat{\boldsymbol{\beta}}_n(k); W) &= \sigma_{zz}tr[W(R(k)(C^{-1} - A)R(k)')] \\
&\quad + \left[kC^{-1}(k)\boldsymbol{\beta} + R(k)\boldsymbol{\delta}\right]' W \left[kC^{-1}(k)\boldsymbol{\beta} + R(k)\boldsymbol{\delta}\right].
\end{aligned}$$

(iii)

$$\begin{aligned}
b_3(\hat{\boldsymbol{\beta}}_n^{PT}(k)) &= -kC^{-1}(k)\boldsymbol{\beta} - R(k)\boldsymbol{\delta}\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2], \\
M_3(\hat{\boldsymbol{\beta}}_n^{PT}(k)) &= \sigma_{zz}[R(k)C^{-1}R(k)'] - \sigma_{zz}[R(k)AR(k)']\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \\
&\quad [R(k)\boldsymbol{\delta}\boldsymbol{\delta}'R(k)']\{2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\} \\
&\quad + k^2[C^{-1}(k)\boldsymbol{\beta}\boldsymbol{\beta}'C^{-1}(k)] \\
&\quad + k[R(k)\boldsymbol{\delta}\boldsymbol{\beta}'C^{-1}(k) + C^{-1}(k)\boldsymbol{\beta}\boldsymbol{\delta}'R(k)]\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2], \\
R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k); W) &= \sigma_{zz}tr[W\left(R(k)C^{-1}R(k)'\right)] - \sigma_{zz}tr[W(R(k)AR(k)')]\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \\
&\quad + [\boldsymbol{\delta}'R(k)WR(k)\boldsymbol{\delta}][2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]] \\
&\quad + k^2[\boldsymbol{\beta}'C^{-1}(k)WR^{-1}(k)\boldsymbol{\beta}] \\
&\quad + 2k[\boldsymbol{\delta}'R(k)'WC^{-1}(k)\boldsymbol{\beta}]\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2].
\end{aligned}$$

(iv)

$$\begin{aligned}
b_4(\hat{\boldsymbol{\beta}}_n^{S}(k)) &= -kC^{-1}(k)\boldsymbol{\beta} + (q-2)R(k)\boldsymbol{\delta}E\left[\chi_{q+2}^{-2}(\Delta^2)\right], \\
M_4(\hat{\boldsymbol{\beta}}_n^{S}(k)) &= \sigma_{zz}\left[R(k)\Sigma_{XX}^{-1}R(k)'\right] - \sigma_{zz}(q-2)\left[R(k)AR(k)'\right]\left\{2E\left[\chi_{q+2}^{-2}(\Delta^2)\right]\right. \\
&\quad \left. -(q-2)E\left[\chi_{q+2}^{-4}(\Delta^2)\right]\right\} + (q^2-4)\left[R(k)\boldsymbol{\delta}\boldsymbol{\delta}'R(k)'\right]E\left[\chi_{q+4}^{-4}(\Delta^2)\right] \\
&\quad + k^2C^{-1}(k)\boldsymbol{\beta}\boldsymbol{\beta}'C^{-1}(k) \\
&\quad + k\left[R(k)\boldsymbol{\delta}\boldsymbol{\beta}'C^{-1}(k) + C^{-1}(k)\boldsymbol{\beta}\boldsymbol{\delta}'R(k)'\right]E\left[\chi_{q+2}^{-2}(\Delta^2)\right], \\
R_4(\hat{\boldsymbol{\beta}}_n^{S}(k); W) &= \sigma_{zz}tr\left(W\left[R(k)\Sigma_{XX}^{-1}R(k)'\right]\right) \\
&\quad - \sigma_{zz}(q-2)tr\left(W\left[R(k)AR(k)'\right]\right)\left\{2E\left[\chi_{q+2}^{-2}(\Delta^2) - (q-2)\chi_{q+2}^{-4}(\Delta^2)\right]\right\} \\
&\quad + (q^2-4)tr\left[W\left(R(k)\boldsymbol{\delta}\boldsymbol{\delta}'R(k)'\right)\right]E\left[\chi_{q+4}^{-4}(\Delta^2)\right] \\
&\quad + k^2\boldsymbol{\beta}'C^{-1}(k)WC^{-1}(k)\boldsymbol{\beta} \\
&\quad + k\ tr\left\{W\left[R(k)\boldsymbol{\delta}\boldsymbol{\beta}'C^{-1}(k) + C^{-1}(k)\boldsymbol{\beta}\boldsymbol{\delta}'R(k)'\right]\right\}E\left[\chi_{q+2}^{-2}(\Delta^2)\right].
\end{aligned}$$

(v)

$$
\begin{aligned}
b_5(\hat{\boldsymbol{\beta}}_n^{S+}(k)) &= b_4(\hat{\boldsymbol{\beta}}_n^{S}(k)) \\
&\quad -R(k)\boldsymbol{\delta}\left\{\mathcal{H}_{q+2}-(q-2)E\left[\chi_{q+2}^{-2}(\Delta^2)I\left(\chi_{q+2}^{-2}(\Delta^2)<(q-2)\right)\right]\right\}, \\
M_5(\hat{\boldsymbol{\beta}}_n^{S+}(k)) &= M_4(\hat{\boldsymbol{\beta}}_n^{S}(k)) \\
&\quad -\sigma_{zz}\left[R(k)AR(k)'\right]E\left[\left(1-(q-2)\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^{2}(\Delta^2)<(q-2)\right)\right] \\
&\quad -\left[R(k)\boldsymbol{\delta}\boldsymbol{\delta}'R(k)'\right]\left\{2E\left[\left(1-(q-2)\chi_{q+2}^{-2}(\Delta^2)\right)I\left(\chi_{q+2}^{-2}(\Delta^2)<(q-2)\right)\right]\right. \\
&\quad \left.-E\left[\left(1-(q-2)\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^{-2}(\Delta^2)<(q-2)\right)\right]\right\} \\
&\quad \times\left\{R(k)\boldsymbol{\delta}\boldsymbol{\beta}'C^{-1}(k)+C^{-1}(k)\boldsymbol{\beta}\boldsymbol{\delta}'R(k)'\right\} \\
&\quad \times E\left[\left(1-(q-2)\chi_{q+2}^{-2}(\Delta^2)\right)I\left(\chi_{q+4}^{2}(\Delta^2)<(q-2)\right)\right], \\
R_5(\hat{\boldsymbol{\beta}}_n^{S+}(k);W) &= R_4(\hat{\boldsymbol{\beta}}_n^{S}(k);W) \\
&\quad -\left\{\sigma_{zz}tr\left(R(k)AR(k)'\right)E\left[\left(1-(q-2)\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^{2}(\Delta^2)<(q-2)\right)\right]\right\} \\
&\quad +\left(\boldsymbol{\delta}'W^2\boldsymbol{\delta}\right)\left\{2E\left[\left(1-(q-2)\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^{2}(\Delta^2)<(q-2)\right)\right]\right. \\
&\quad \left.-E\left[\left(1-(q-2)\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^{2}(\Delta^2)<(q-2)\right)\right]\right\} \\
&\quad +2k\boldsymbol{\delta}'R(k)'WC^{-1}(k)\boldsymbol{\beta}E\left[\left(1-(q-2)\chi_{q+2}^{-2}(\Delta^2)\right)I\left(\chi_{q+2}^{2}(\Delta^2)<(q-2)\right)\right].
\end{aligned}
$$

4 Comparison of Estimators of $\boldsymbol{\beta}$

First note that the asymptotic covariance matrix of unrestricted estimator of $\boldsymbol{\beta}$ is $\sigma_{zz}C^{-1}$ where $\Leftrightarrow$ $C = K'_{xx}\Sigma_{XX}K_{xx}$ is positive definite matrix. Thus we can find an orthogonal matrix Γ such that

$$\Gamma'\left(K'_{xx}\Sigma_{XX}K_{xx}\right)\Gamma = \Gamma'C\Gamma = diag(\lambda_1,\lambda_2,\ldots,\lambda_p), \tag{4.1}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots, \lambda_p > 0$ are the characteristic roots of the matrix $(K'_{xx}\Sigma_{XX}K_{xx})$. It is easy to see that the characteristic roots of $\left[I_p + k\left(K'_{xx}\Sigma_{XX}K_{xx}\right)^{-1}\right]^{-1} = R(k)$ and of $[(K'_{xx}\Sigma_{XX}K_{xx}) + kI_p] = R^{-1}(k)$ are

$$\left(\frac{\lambda_1}{(\lambda_1+k)^2},\frac{\lambda_2}{(\lambda_2+k)^2},\ldots,\frac{\lambda_p}{(\lambda_p+k)^2}\right) \text{ and } (\lambda_1+k,\lambda_2+k,\ldots,\lambda_p+k) \tag{4.2}$$

respectively. Hence we obtain the following identities:

$$
\begin{aligned}
tr[R(k)C^{-1}R(k)'] &= tr\left[R(k)\left(K'_{xx}\Sigma_{XX}K_{xx}\right)^{-1}R(k)'\right] = \sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)}, && (4.3)\\
\boldsymbol{\beta}'R^{-2}(k)\boldsymbol{\beta} &= \sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+k)^2},\ \boldsymbol{\theta}=\Gamma'\boldsymbol{\beta}=(\theta_1,\theta_2,\ldots,\theta_p)', && (4.4)\\
tr[R(k)AR(k)'] &= \sum_{i=1}^{p}\frac{a_{ii}^*\lambda_i^2}{(\lambda_i+k)^2} && (4.5)
\end{aligned}
$$

where $a_{ii}^* \geq 0$ is the diagonal matrix of $A^* = \Gamma' A \Gamma$ and

$$tr\left[\boldsymbol{\delta}'R(k)'R(k)\boldsymbol{\delta}\right] = \sum_{i=1}^{p}\frac{\lambda_i\delta_i^{*2}}{(\lambda_i+k)^2} \tag{4.6}$$

where δ_i^* is the i^{th} element of $\boldsymbol{\delta}^* = \boldsymbol{\delta}'\Gamma$. Similarly

$$tr\left[\boldsymbol{\delta}'R(k)'\left(K'_{xx}\Sigma_{XX}K_{xx}+kI_p\right)^{-1}R(k)\boldsymbol{\delta}\right] = \sum_{i=1}^{p}\frac{\theta_i\lambda_i\delta_i^*}{(\lambda_i+k)^2}. \tag{4.7}$$

4.1 Comparison of $\tilde{\boldsymbol{\beta}}_n(k)$ and $\tilde{\boldsymbol{\beta}}_n$:

In this case we have the risk of $\tilde{\boldsymbol{\beta}}_n(k)$ as

$$R(\tilde{\boldsymbol{\beta}}_n(k); I_p) = \sigma_{zz}\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2} + k^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+k)^2}. \tag{4.8}$$

Clearly for $k=0$, the risk equals that the risk of $\tilde{\boldsymbol{\beta}}_n$. Note that the first term of (4.8) is a continuous, monotonically decreasing function of k and its derivative with respect to k approaches $-\infty$ as $k \to 0^+$ and $\lambda_p \to 0$. The second term is also a continuous monotonically increasing function of k and its derivative with respective k tends to zero as $k \to 0^+$ and the second term approaches $\boldsymbol{\beta}\boldsymbol{\beta}'$ as $k \to \infty$. Differentiating with respect to k, we get

$$\frac{\partial R(\tilde{\boldsymbol{\beta}}_n(k); I_p)}{\partial k} = 2\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^3}(k\theta_i^2 - \sigma_{zz}). \tag{4.9}$$

Thus a sufficient condition for (4.9) to be negative is that $0 < k < k_0^*$ where

$$k_0^* = \frac{\sigma_{zz}}{\theta_{\max}}, \tag{4.10}$$

where $\theta_{\max}$ = Largest element of $\boldsymbol{\theta}$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)'$.

Thus

$$R_1(\tilde{\boldsymbol{\beta}}_n; I_p) \geq R_1(\tilde{\boldsymbol{\beta}}_n(k); I_p) \ \text{ for } \ 0 < k \leq k_0^*.$$

4.2 Comparison of $\hat{\boldsymbol{\beta}}_n(k)$ and $\hat{\boldsymbol{\beta}}_n$:

In this case, assume $H\boldsymbol{\beta} = \boldsymbol{h}$, then

$$\begin{aligned} R_2(\hat{\boldsymbol{\beta}}_n(k); I_p) &= \sigma_{zz}\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2} - \sigma_{zz}\sum_{i=1}^{p}\frac{a_{ii}^*}{(\lambda_i+k)^2} + k^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+k)^2} \\ &= \sigma_{zz}\sum_{i=1}^{p}\frac{(\lambda_i - a_{ii}^*)}{(\lambda_i+k)^2} + \sum_{i=1}^{p}\frac{\theta_i^2}{(1+\frac{\lambda_i}{k})^2}. \end{aligned} \tag{4.11}$$

where $a_{ii}^* \geq 0$ is the diagonal element of $\boldsymbol{\delta}^* = \Gamma'\boldsymbol{\delta}$.

Differentiating with respect to k, we have

$$\frac{\partial R_2(\hat{\boldsymbol{\beta}}_n(k); I_p)}{\partial k} = 2\sum_{i=1}^{p}\frac{[k\theta_i^2\lambda_i - \sigma_{zz}(\lambda_i - a_{ii}^*)]}{(\lambda_i+k)^2}. \tag{4.12}$$

It is obvious that a sufficient condition for the derivative (4.12) is negative is that $0 < k < k_1^*$ where

$$k_1^* = \frac{\sigma_{zz}\min_{1\leq i\leq p}(\lambda_i - a_{ii}^*)}{\max_{1\leq j\leq p}(\theta_i^2\lambda_i)}. \tag{4.13}$$

It is clear that $k_1^* \leq k_0^*$ since $\frac{\lambda_i - a_{ii}^*}{\lambda_i} \leq 1$ for all $a_{ii}^* \geq 0$ and $\lambda_i^* > 0, i = 1, 2, \ldots p$. Hence the range of values of k for the dominance of $\hat{\boldsymbol{\beta}}_n(k)$ over $\hat{\boldsymbol{\beta}}_n$ is smaller than the dominance of $\tilde{\boldsymbol{\beta}}_n(k)$ over $\tilde{\boldsymbol{\beta}}_n$. Thus, one concludes that the restricted ridge regression estimator has smaller risk value than the unrestricted ridge regression estimator under $H\boldsymbol{\beta} = \boldsymbol{h}$.

Now under the alternative hypothesis $H\boldsymbol{\beta} \neq \boldsymbol{h}$, we may write

$$R_2(\hat{\boldsymbol{\beta}}_n(k); I_p) = \sum_{i=1}^{p}\frac{1}{(\lambda_i+k)^2}\left[\sigma_{zz}(\lambda_i - a_{ii}^*) + k^2\theta_i^2 + \lambda_i^2\delta_i^{*2} + 2k\theta_i\delta_i^*\lambda_i\right]. \tag{4.14}$$

Thus differentiating (4.14) with respect to k, we obtain a sufficient condition for $\partial R_2(\hat{\boldsymbol{\beta}}_n(k); I_p)/\partial k$ to be negative as $k \in (0, k_2^*)$ where

$$k_2^* = \frac{\min_{1\leq i\leq p}\left[(\lambda_i - a_{ii}^*) - \lambda_i^2\delta_i^*(\theta_i - \delta_i^*)\right]}{\max_{1\leq i\leq p}\lambda_i\theta_i(\theta_i - \delta_i^*)}. \tag{4.15}$$

Thus a sufficient condition for the restricted ridge regression estimator to have smaller risk value than the unrestricted ridge regression estimator is that there exists a value of k such that $0 < k < k_1$ where k_1 is given by

$$k_1 = \frac{\min_{1\leq i\leq p}\left[\sigma_{zz}a_{ii}^* - \lambda_i^2\delta_i^{*2}\right]}{\max_{1\leq i\leq p}(2\theta_i\delta_i^*\lambda_i)}. \tag{4.16}$$

We conclude that

$$R_1(\tilde{\boldsymbol{\beta}}_n(k); I_p) - R_2(\hat{\boldsymbol{\beta}}_n(k); I_p) \geq 0 \text{ for all } k \text{ such that } 0 < k < k_1 ,$$

since

$$R_1(\tilde{\boldsymbol{\beta}}_n(k); I_p) - R_2(\hat{\boldsymbol{\beta}}_n(k); I_p) = \sum_{i=1}^{p} \frac{(\sigma_{zz} a_{ii}^* - \lambda_i^2 \delta_i^{*2} - 2k\theta_i \lambda_i \delta_i^*)}{(\lambda_i + k)^2}.$$

4.3 Comparison of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ and $\hat{\boldsymbol{\beta}}_n^{PT}$:

Under $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$, we obtain

$$R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k); I_p) = \sigma_{zz} \sum_{i=1}^{p} \frac{[\lambda_i - a_{ii}^* \mathcal{H}_{q+2}(\chi_q^2(\alpha); 0)]}{(\lambda_i + k)^2} + k^2 \sum_{i=1}^{p} \frac{\theta_i^2}{(\lambda_i + k)^2}. \tag{4.17}$$

Then, the derivative with respect to k, we have

$$\frac{\partial R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k); I_p)}{\partial k} = 2 \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \left\{ k\theta_i^2 \lambda_i - \sigma_{zz} \left[\lambda_i - a_{ii}^* \mathcal{H}_{q+2}(\chi_q^2(\alpha); 0) \right] \right\}. \tag{4.18}$$

A sufficient condition for (4.18) to be negative is that $k \in (0, k_3^*)$ where

$$k_3^*(\alpha) = \frac{\sigma_{zz} \min_{1 \leq i \leq p} \left[\lambda_i - a_{ii}^* \mathcal{H}_{q+2}(\chi_q^2(\alpha); 0) \right]}{\max_{1 \leq i \leq p}(\theta_i^2 \lambda_i)}, \quad 0 \leq \alpha \leq 1. \tag{4.19}$$

Note that if $\alpha = 0$, then $R_1(\tilde{\boldsymbol{\beta}}_n(k); I_p) < R_1(\tilde{\boldsymbol{\beta}}_n; I_p)$ under $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$ while for $\alpha = 1$, $R_1(\hat{\boldsymbol{\beta}}_n(k); I_p) < R_2(\hat{\boldsymbol{\beta}}_n; I_p)$.

When the null hypothesis does not hold, we consider the risk difference

$$\begin{aligned} R_3(\hat{\boldsymbol{\beta}}_n^{PT}; I_p) - R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k); I_p) \;\; = \;\; & \sigma_{zz} tr \left[C^{-1} - R(k) C^{-1} R(k)' \right] \\ & -\sigma_{zz} tr \left[A - R(k) A R(k)' \right] \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \\ & +\boldsymbol{\delta}' \left[I_p - R(k)^2 \right] \boldsymbol{\delta} \left\{ 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \right. \\ & \left. -\mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2] \right\} \\ & -2k \left[\boldsymbol{\delta}' R(k)' C^{-1} R(k) \boldsymbol{\beta} \right] \mathcal{H}_q(\chi_q^2(\alpha); \Delta^2) \\ & -k^2 \boldsymbol{\beta}' C^{-2}(k) \boldsymbol{\beta}. \end{aligned} \tag{4.20}$$

The R.H.S. of (4.20)≥ 0 whenever

$$\boldsymbol{\delta}'[I_p - R(k)^2]\boldsymbol{\delta} \geq \frac{f_1(k, \alpha, \Delta^2)}{[2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]]}, \tag{4.21}$$

where

$$f_1(k,\alpha,\Delta^2) = \sigma_{zz} tr\left[A - R(k)AR(k)'\right]\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] \quad (4.22)$$
$$-\sigma_{zz} tr\left[C^{-1} - R(k)C^{-1}R(k)'\right]$$
$$+2k\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta} + k^2\boldsymbol{\beta}'C^{-2}(k)\boldsymbol{\beta}.$$

Thus it can be shown by Courant-Fisher theorem (Rao (1973), pp. 48-53)) that $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ is superior to $\hat{\boldsymbol{\beta}}_n^{PT}$ with respect to the criterion of risk, whenever $\Delta_1^2(\alpha) \leq \Delta^2 < \infty$, where

$$\Delta_1^2(\alpha) = \frac{f_1(k,\alpha,\Delta^2)}{Ch_{\max}\left[\{I - R^2(k)\}C^{-1}\right]\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] - \mathcal{H}_{q+4}(\chi_q^2(\alpha);\Delta^2]\right]} \quad (4.23)$$

and $Ch_{\max}(M)$ is the largest characteristic root of the matrix M.

Now we consider the $R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k), I_p)$ which as a function of eigenvalues and k is given as follows:

$$R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k), I_p) = \sum_{i=1}^{p}\frac{1}{(\lambda_i+k)^2}\Big[\sigma_{zz}\left\{\lambda_i - a_{ii}^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\right\} + k^2\theta_i^2 \quad (4.24)$$
$$+2k\theta_i\lambda_i\delta_i^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]$$
$$+\lambda_i^2\delta_i^{*2}\left\{2\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha);\Delta^2]\right\}\Big].$$

Differentiating it with respect to k, we obtain

$$\frac{\partial R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k), I_p)}{\partial k} = 2\sum_{i=1}^{p}\frac{1}{(\lambda_i+k)^3}\Big[k\lambda_i\theta_i\left\{\theta_i - \delta_i^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\right\} \quad (4.25)$$
$$-\Big\{\sigma_{zz}(\lambda_i - a_{ii}^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2])$$
$$+\lambda_i^2\delta_i^{*2}\left\{2\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha);\Delta^2]\right\}$$
$$-\theta_i\lambda_i^2\delta_i^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\Big\}\Big].$$

Hence we find a sufficient condition on k such that $0 < k < k_2(\alpha,\Delta^2)$ so that the derivative $\partial R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k), I_p)/\partial k$ is negative where $k_2(\alpha,\Delta^2)$ is defined by

$$k_2(\alpha,\Delta^2) = \frac{f_2(\alpha,\Delta^2)}{g_1(\alpha,\Delta^2)} \quad (4.26)$$

where

$$f_2(\alpha,\Delta^2) = \min_{1\leq i\leq p}\Big[\sigma_{zz}\left(\lambda_i - a_{ii}^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\right) + \lambda_i^2\delta_i^{*2}\Big\{2\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]$$
$$-\mathcal{H}_{q+4}[\chi_q^2(\alpha);\Delta^2]\Big\} - \theta_i\lambda_i^2\delta_i^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\Big], \quad (4.27)$$

and

$$g_1(\alpha, \Delta^2) = \max_{1\leq i\leq p} \left[\lambda_i\theta_i \left\{\theta_i - \delta_i^* \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right\}\right]. \tag{4.28}$$

Suppose $k > 0$, then the following statements hold true following Kaciranlar, Sakallioglu, Akdeniz, Styan and Werner (1999):

1. If $g_1(\alpha, \Delta^2) > 0$, it follows that for each $k > 0$ with $k < k_2(\alpha, \Delta^2)$, $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n^{PT}$.

2. If $g_1(\alpha, \Delta^2) < 0$, it follows that for each $k > 0$ with $k > k_2(\alpha, \Delta^2)$, $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n^{PT}$.

4.4 Comparison of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$, $\tilde{\boldsymbol{\beta}}_n(k)$ and $\hat{\boldsymbol{\beta}}_n(k)$:

Consider first the comparison of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ and $\hat{\boldsymbol{\beta}}_n(k)$.

Under the alternative hypothesis $H\boldsymbol{\beta} \neq \boldsymbol{h}$, the difference between the risks of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ and $\hat{\boldsymbol{\beta}}_n(k)$ is

$$\begin{aligned} &R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k) : I_p) - R_2(\hat{\boldsymbol{\beta}}_n(k); I_p) \\ &= \sigma_{zz} tr\left[R(k)AR(k)'\right]\left[1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right] \\ &- (\boldsymbol{\delta}' R(k)' R(k)\boldsymbol{\delta})\left[1 - 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right] \\ &-2k\boldsymbol{\delta}' R(k)' R^{-1}(k)\boldsymbol{\beta}\left[1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right]. \end{aligned} \tag{4.29}$$

The expression is nonnegative whenever

$$\begin{aligned} &\boldsymbol{\delta} R(k)' R(k)\boldsymbol{\delta} \geq \\ &\frac{\sigma_{zz} tr\left[R(k)AR(k)'\right]\left[1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right] - 2k\boldsymbol{\delta}' R(k)' C^{-1}(k)\boldsymbol{\beta}\left[1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right]}{\left[1 - 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right]}. \end{aligned} \tag{4.30}$$

Using the Courant-Fisher theorem, we obtain that the right hand side of (4.30) is nonnegative according as

$$\Delta_2^2(\alpha, k) \geq \frac{\left\{\sigma_{zz} tr\left[R(k)AR(k)'\right] - 2k\boldsymbol{\delta}' R(k)' C^{-1}(k)\boldsymbol{\beta}\right\}\left[1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right]}{Ch_{\max}\left(R(k)' R(k) C^{-1}\right)\left[1 - 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right]}. \tag{4.31}$$

Thus $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ is dominated by $\hat{\boldsymbol{\beta}}_n(k)$ whenever $\Delta^2 \in (0, \Delta_2^2(\alpha, k))$. When $\alpha = 1$, then $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ is dominated by $\hat{\boldsymbol{\beta}}_n(k)$ whenever $\Delta^2 \in (0, \Delta_3^2(1))$, where

$$\Delta_3^2(1) = \frac{\{\sigma_{zz} tr\,[R(k)AR(k)'] - 2k\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta}\}}{Ch_{\max}\,(R(k)'R(k)C^{-1})}. \tag{4.32}$$

Rewriting the expression (4.30) in terms of eigenvalues and k, we obtain

$$\begin{aligned} R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k); I_p) - R_2(\hat{\boldsymbol{\beta}}_n(k); I_p) &= \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \big[\sigma_{zz} a_{ii}^* \left[1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right] \\ &\quad -\lambda_i^2 \delta_i^{*2} \left[1 - 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right] \\ &\quad -2k\theta_i\lambda_i\delta_i \left[1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right]\big] \end{aligned}$$

and the right hand side is negative when

$$k_3(\Delta^2, \alpha) = \frac{\max_{1\le i\le p}\left\{\left[\sigma_{zz}a_{ii}^*\left[1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right] - \lambda_i^2\delta_i^{*2}\left[1 - 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right]\right]\right\}}{\min_{1\le i\le p}\left\{2\theta_i\lambda_i\delta_i^*\left[1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right]\right\}}.$$

Thus $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ dominates $\hat{\boldsymbol{\beta}}_n(k)$ whenever $k_3(\Delta^2, \alpha) < k$, otherwise the reverse holds true. For $\alpha = 1$, we find that $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ is dominated by $\hat{\boldsymbol{\beta}}_n(k)$ when

$$k_4(\Delta^2, 1) = \frac{\max_{1\le i\le p}\left[\sigma_{zz}a_{ii}^* - \lambda_i^2\delta_i^{*2}\right]}{\min_{1\le i\le p}(2\theta_i\lambda_i\delta_i^*)}. \tag{4.33}$$

Under the null hypothesis $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$, $\hat{\boldsymbol{\beta}}_n(k)$ is superior to $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ since the risk difference equals

$$\sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2}\left[\sigma_{zz}a_{ii}^*\{1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); 0]\}\right] \ge 0.$$

Now we consider the comparison between the risks of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ and $\tilde{\boldsymbol{\beta}}_n(k)$ as follows:

$$\begin{aligned} R_3(\hat{\boldsymbol{\beta}}_n^{PT}; I_p) - R_1(\tilde{\boldsymbol{\beta}}_n(k); I_p) &= \sigma_{zz} tr\,[R(k)AR(k)']\,\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \\ &\quad - [\boldsymbol{\delta}'R(k)'R(k)\boldsymbol{\delta}]\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right] \\ &\quad -2k\left[\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta}\right]\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]. \end{aligned}$$

The expression on the right hand side is nonnegative whenever

$$\boldsymbol{\delta}'R(k)'R(k)\boldsymbol{\delta} \ge \frac{\{\sigma_{zz} tr\,[R(k)AR(k)'] - 2k\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta}\}\,\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]}{\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right]}. \tag{4.34}$$

The use of Courant-Fisher theorem once again yields that (4.34) is nonnegative according to

$$\Delta_4^2(\alpha,k) \geq \frac{\{\sigma_{zz} tr\,[R(k)AR(k)'] - 2k\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta}\}\,\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]}{Ch_{\max}\left(R(k)'R(k)C^{-1}\right)\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha);\Delta^2]\right]}. \tag{4.35}$$

Thus $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ is dominated by $\hat{\boldsymbol{\beta}}_n(k)$ whenever $\Delta^2 \in (0, \Delta_4^2(\alpha,k))$.

Rewriting the expression in (4.34) in terms of eigenvalues and k, we obtain

$$\begin{aligned} R_3(\hat{\boldsymbol{\beta}}_n^{PT}(k);I_p) - R_1(\tilde{\boldsymbol{\beta}}_n(k);I_p) \;=\; & \sum_{i=1}^{p}\frac{1}{(\lambda_i+k)^2}\left[\sigma_{zz}\lambda_i^2 a_{ii}^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\right. \\ & -\lambda_i^2\delta_i^{*2}\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha);\Delta^2]\right] \\ & \left.-2k\theta_i\lambda_i\delta_i^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\right] \end{aligned} \tag{4.36}$$

and the right hand side is positive meaning thereby that $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ dominates $\tilde{\boldsymbol{\beta}}_n(k)$ when $k_5(\Delta^2,\alpha) < k$, where

$$\begin{aligned} & k_5(\Delta^2,\alpha) = \\ & \frac{\max_{1\leq i\leq p}\left\{\left[\sigma_{zz}\lambda_i^2 a_{ii}^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] - \lambda_i^2\delta_i^{*2}\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha);\Delta^2]\right]\right]\right\}}{\min_{1\leq i\leq p}\left\{2k\theta_i\lambda_i\delta_i^*\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\right\}} \end{aligned} \tag{4.37}$$

and $\tilde{\boldsymbol{\beta}}_n(k)$ dominates $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ whenever $k_5(\alpha,\Delta^2) > k$.

Now we consider the relative efficiency (RE) of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ compared with $\tilde{\boldsymbol{\beta}}_n(k)$. Accordingly, we provide a maximum and minimum (Max and Min) rule for the optimum choice of the level of significance of the $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ for testing the null hypothesis $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$, . For fixed value of $k (k > 0)$, this RE is a function of α and Δ^2. Let us denote this by

$$E(\alpha,\Delta,k) = \frac{R(\tilde{\boldsymbol{\beta}}_n^{R}(k);I_p)}{R(\hat{\boldsymbol{\beta}}_n^{PT}(k);I_p)} = \left[1 - \frac{f_{14}(\alpha,k,\Delta^2)}{\sigma_{zz}tr[R(k)AR(k)'] + k^2\boldsymbol{\beta}'C^{-2}(k)'\boldsymbol{\beta}}\right]^{-1} \tag{4.38}$$

where

$$\begin{aligned} f_{14}(\alpha,k,\Delta^2) \;=\; & \sigma_{zz}tr[R(k)AR(k)']\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] \\ & -\boldsymbol{\delta}'R(k)'R(k)\boldsymbol{\delta}\left\{2\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha);\Delta^2]\right\} \\ & -2k\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta}. \end{aligned} \tag{4.39}$$

For a given k, the function $E(\alpha,\Delta^2,k)$, is a function of α and Δ^2. This function for $\alpha \neq 0$ has its maximum under the null hypothesis with following value,

$$E_{\max}(\alpha,0,k) = \left[1 - \frac{tr[R(k)AR(k)']\mathcal{H}_{q+2}[\chi_q^2(\alpha);0]}{tr[R(k)AR(k)'] + k^2\boldsymbol{\beta}'C^{-2}(k)'\boldsymbol{\beta}}\right]^{-1}. \tag{4.40}$$

For given k, $E_{\max}(\alpha, 0, k)$ is a decreasing function of α. While, the minimum efficiency $E_{\min}$ is an increasing function of α. For $\alpha \neq 0$, as Δ^2 varies the graphs of $E(0, \Delta, k)$ and $E(1, \Delta, k)$ intersects in the range $0 < \Delta^2 < \Delta_2^2(\alpha, k)$, which is given in (4.32). Therefore, in order to choose an estimator with optimum relative efficiency, we adopt the following rule for fixed values of k. If $0 < \Delta^2 < \Delta_2^2(\alpha, k)$, we choose $\hat{\boldsymbol{\beta}}_n^R(k)$ since $E(0, \Delta, k)$ is the largest in this interval. However, Δ^2 is unknown and there is no way of choosing a uniformly best estimator. Therefore, following Saleh (2006), we will use the following criterion for selecting the significance level of the preliminary test.

Suppose the experimenter does not know the size of α and wants an estimator which has relative efficiency not less than $E_{\min}$. Then among the set of estimators with $\alpha \in A$, where $A = \{\alpha : E(\alpha, \Delta, k) \geq E_{\min} \text{for all} \Delta\}$, the estimator is chosen to maximizes $E(\alpha, \Delta, k)$ over all $\alpha \in A$ and all Δ^2. Thus we solve for α from the following equation.

$$\max_{0 \leq \alpha \leq 1} \min_{\Delta^2} E(\alpha, \Delta, k) = E_{\min}. \tag{4.41}$$

The solution α^* for (4.41) gives the optimum choice of α and the value of $\Delta^2 = \Delta^2_{\min}$ for which (4.40) is satisfied. The maximum and minimum guaranteed efficiencies for $p = 6, q = 4, \sigma_{zz} = 1$ and different values of k is presented in Table 1. We observe that the maximum guaranteed efficiency is decreasing function of α the size of the test where as the minimum guaranteed efficiency is a increasing function of α.

Suppose for example, $p = 6, q = 4, \sigma_{zz} = 1$ and $k = 0.01$ and the experimenter wishes to have an estimator with a minimum guaranteed efficiency of 0.75. Note that when $p = 6, q = 4, \sigma_{zz} = 1$ and $k = 0.50$, the size of α is 0.15 with a minimum guaranteed efficiency 0.7493 and maximum efficiency of 1.7216. Thus with these conditions, we choose $\alpha^* = min(0.05, 0.15) = 0.05$ which corresponds to $k = 0.01$.

Note that when $p = 6, q = 4, k = 0.01$ and $\sigma_{zz} = 1$, the size of α is 0.50 with a minimum guaranteed efficiency 0.9660 and maximum efficiency of 1.1877. Thus with these conditions, we choose $\alpha^* = \min(0.01, 0.50) = 0.01$ which corresponds to $\sigma_{zz} = 2$.

Table 1: Maximum and minimum guaranteed efficiency of PTRRRE for $p = 6$, $q = 4$ and for selected values of k, σ_{zz} and α.

k	σ_{zz}	α	1%	5%	10%	15%	20%	25%	30%	50%
0.01	1.0	E_{Max}	2.7836	2.3149	1.9875	1.7749	1.6225	1.4158	1.2813	1.1877
		E_{Min}	0.6265	0.7551	0.8190	0.8578	0.8854	0.9232	0.9483	0.9660
		Δ_{Min}	10.9589	8.6758	7.6484	7.0776	6.7352	6.1644	5.7078	5.3653
0.01	1.25	E_{Max}	2.7837	2.3150	1.9875	1.7750	1.6225	1.4158	1.2813	1.1877
		E_{Min}	0.7663	0.8646	0.9066	0.9298	0.9453	0.9652	0.9775	0.9857
		Δ_{Min}	12.1005	9.9315	9.0183	8.4475	7.9909	7.4201	7.0776	6.7352
0.01	1.5	E_{Max}	2.7837	2.3150	1.9875	1.775	1.6226	1.4158	1.2814	1.1877
		E_{Min}	0.8628	0.9293	0.9541	0.9669	0.9750	0.9848	0.9905	0.9941
		Δ_{Min}	13.6986	11.5297	10.6164	10.0457	9.5890	9.0183	8.5616	8.2192
0.01	2.0	E_{Max}	2.7837	2.3150	1.9875	1.7750	1.6226	1.4158	1.2814	1.1877
		E_{Min}	0.9609	0.9838	0.9907	0.9937	0.9955	0.9975	0.9986	0.9992
		Δ_{Min}	17.5799	15.5251	14.6119	14.0411	13.6986	13.0137	12.5571	12.1005
0.10	1.0	E_{Max}	2.7755	2.3099	1.9843	1.7727	1.6209	1.4148	1.2808	1.1873
		E_{Min}	0.5847	0.7210	0.7914	0.8349	0.8663	0.9098	0.9390	0.9597
		Δ_{Min}	10.6164	8.3333	7.4201	6.8493	6.5068	5.9361	5.4795	5.137
0.10	1.25	E_{Max}	2.7784	2.3118	1.9854	1.7735	1.6215	1.4151	1.281	1.1875
		E_{Min}	0.7280	0.8383	0.8871	0.9146	0.9332	0.9572	0.9722	0.9822
		Δ_{Min}	11.6438	9.4749	8.5616	7.9909	7.6484	7.0776	6.6210	6.3927
0.10	1.5	E_{Max}	2.7801	2.3127	1.9861	1.7739	1.6218	1.4153	1.2811	1.1876
		E_{Min}	0.8321	0.9110	0.9415	0.9575	0.9678	0.9803	0.9877	0.9923
		Δ_{Min}	13.0137	10.9589	10.0457	9.4749	9.1324	8.5616	8.1050	7.7626
0.10	2.0	E_{Max}	2.7817	2.3137	1.9867	1.7744	1.6221	1.4155	1.2812	1.1876
		E_{Min}	0.9460	0.9770	0.9865	0.9909	0.9935	0.9964	0.9979	0.9988
		Δ_{Min}	16.5525	14.4977	13.5845	13.1279	12.6712	12.1005	11.6438	11.3014
0.50	1.0	E_{Max}	2.5983	2.1993	1.9120	1.7216	1.5831	1.3926	1.2671	1.1789
		E_{Min}	0.4570	0.6054	0.6918	0.7493	0.7927	0.8561	0.9008	0.9336
		Δ_{Min}	9.7032	7.5342	6.6210	6.0502	5.7078	5.2511	4.7945	4.5662
0.50	1.25	E_{Max}	2.6605	2.2385	1.9378	1.7399	1.5967	1.4006	1.2721	1.1820
		E_{Min}	0.5953	0.7360	0.8068	0.8497	0.8800	0.9211	0.9478	0.9662
		Δ_{Min}	10.3881	8.3330	7.4201	6.8493	6.5068	6.0502	5.7078	5.3653
0.50	1.5	E_{Max}	2.6963	2.2609	1.9525	1.7503	1.6044	1.4051	1.2748	1.1837
		E_{Min}	0.7115	0.8305	0.8831	0.9126	0.9323	0.9574	0.9728	0.9829
		Δ_{Min}	11.3014	9.2466	8.3333	7.8767	7.5342	6.9635	6.6210	6.3927
0.50	2.0	E_{Max}	2.7335	2.2840	1.9675	1.7609	1.6122	1.4097	1.2776	1.1854
		E_{Min}	0.8704	0.9377	0.9614	0.9731	0.9802	0.9886	0.9932	0.9959
		Δ_{Min}	13.5845	11.6438	10.8447	10.3881	10.0457	9.5890	9.1324	8.9041

4.5 Comparison of $\hat{\boldsymbol{\beta}}_n^S(k)$ with $(\hat{\boldsymbol{\beta}}_n^S, \tilde{\boldsymbol{\beta}}_n(k)$ and $\hat{\boldsymbol{\beta}}_n^{PT}(k))$:

4.5.1 Comparison of $\hat{\boldsymbol{\beta}}_n^S(k)$ and $\hat{\boldsymbol{\beta}}_n^S$:

Case 1: Under the null hypothesis $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$

Using Theorem 3, the risk function of $\hat{\boldsymbol{\beta}}_n^S(k)$ can be expressed as

$$R_4(\hat{\boldsymbol{\beta}}_n^S(k); I_p) = \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \left[\sigma_{zz}(\lambda_i - (q-2)a_{ii}^*) + k^2\theta_i^2\right]. \tag{4.42}$$

Differentiating (4.42) with respect to k gives

$$\frac{\partial R_4(\hat{\boldsymbol{\beta}}_n^S(k); I_p)}{\partial k} = 2\sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^3} \left[k\theta_i^2 - \sigma_{zz}(\lambda_i - (q-2)a_{ii}^*)\right]. \tag{4.43}$$

A sufficient condition for (4.43) to be negative is that $k \in (0, k_6)$ where

$$k_6 = \frac{\sigma_{zz} \min_{1\le i\le p}\left[\lambda_i - (q-2)a_{ii}^*\right]}{\max_{1\le i\le p} \theta_i^2}. \tag{4.44}$$

Thus $\hat{\boldsymbol{\beta}}_n^S(k)$ dominates $\hat{\boldsymbol{\beta}}_n^S$ for $k \in (0, k_6)$.

Case 2: Under the alternative hypothesis $H_1 : H\boldsymbol{\beta} \neq \boldsymbol{h}$

Using Theorem 3 again, the risk function of $\hat{\boldsymbol{\beta}}_n^S(k)$ can be expressed as

$$\begin{aligned}
&R_4(\hat{\boldsymbol{\beta}}_n^S(k); I_p) \\
&= \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \\
&\times \left[\sigma_{zz}\left\{\lambda_i - (q-2)a_{ii}^*\left[(q-2)E[\chi_{q+2}^{-4}(\Delta^2)] + \left(1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2\sigma_{zz}a_{ii}^*}\right)(2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)]\right]\right\}\right. \\
&\left.+2(q-2)k\theta_i\lambda_i\delta_i^* E[\chi_{q+2}^{-4}(\Delta^2)] + k^2\theta_i^2\right].
\end{aligned} \tag{4.45}$$

Differentiating (4.45) with respect to k gives

$$\begin{aligned}
\frac{\partial R_4(\hat{\boldsymbol{\beta}}_n^S(k); I_p)}{\partial k} &= 2\sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^3} \left\{k\lambda_i\theta_i\left[\theta_i - (q-2)\delta_i^* E[\chi_{q+2}^{-4}(\Delta^2)]\right]\right. \\
&-\sigma_{zz}\left\{\lambda_i - (q-2)a_{ii}^*\left[(q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right.\right. \\
&\left.\left.+\left(1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2\sigma_{zz}a_{ii}^*}\right)(2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)]\right]\right\} \\
&\left.-(q-2)\theta_i\lambda_i^2\delta_i^* E[\chi_{q+2}^{-4}(\Delta^2)]\right\}.
\end{aligned} \tag{4.46}$$

A sufficient condition for (4.46) to be negative is that $k < k_7(\Delta^2)$ where

$$k_7(\Delta^2) = \frac{f_3(\Delta^2)}{g_2(\Delta^2)}, \tag{4.47}$$

where

$$\begin{aligned} f_3(\Delta^2) &= \min_{1\le i\le p} \Big\{\sigma_{zz}\Big\{\lambda_i - (q-2)\sigma_{zz}a^*_{ii}\left[(q-2)E[\chi^{-4}_{q+2}(\Delta^2)]\right. \\ &\quad \left.\left. + \left(1 - \frac{(q+2)\delta^{*2}_i}{2\Delta^2\sigma_{zz}a^*_{ii}}\right)(2\Delta^2)E[\chi^{-4}_{q+4}(\Delta^2)]\right]\right\} \\ &\quad +(q-2)\theta_i\lambda^2_i\delta^*_i E[\chi^{-2}_{q+2}(\Delta^2)]\Big\} \end{aligned} \tag{4.48}$$

and

$$g_2(\Delta^2) = \max_{1\le i\le p}\left[\lambda_i\theta_i\left\{\theta_i - (q-2)\delta^*_i E[\chi^{-4}_{q+2}(\Delta^2)]\right\}\right]. \tag{4.49}$$

Suppose $k > 0$, then the following statements hold true following Kaciranlar, Sakallioglu, Akdeniz, Styan and Werner (1999):

1. If $g_2(\Delta^2) > 0$, it follows that for each $k > 0$ with $k < k_7(\Delta^2)$, $\hat{\boldsymbol{\beta}}^S_n(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}^S_n$.
2. If $g_2(\Delta) < 0$, it follows that for each $k > 0$ with $k > k_7(\Delta)$, $\hat{\boldsymbol{\beta}}^S_n(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}^S_n$.

To find a sufficient condition on Δ^2, consider the difference in the risks of $\hat{\boldsymbol{\beta}}^S_n(k)$ and $\hat{\boldsymbol{\beta}}^S_n$ as

$$\begin{aligned} R(\hat{\boldsymbol{\beta}}^S_n(k); I_p) - R(\hat{\boldsymbol{\beta}}^S_n; I_p) &= \sigma_{zz}\left[tr(R(k)C^{-1}R(k)') - tr(C^{-1})\right] \\ &\quad +(q-2)\sigma_{zz}\left[tr(A) - tr(R(k)AR(k)')\right] \\ &\quad \times\left\{2E[\chi^{-2}_{q+2}(\Delta^2)] - (q-2)E[\chi^{-4}_{q+2}(\Delta^2)]\right\} \\ &\quad -(q^2-4)\boldsymbol{\delta}'[I_p - R(k)'R(k)]\boldsymbol{\delta}E[\chi^{-4}_{q+4}(\Delta^2)] + k^2\boldsymbol{\beta}'C^{-2}(k)\boldsymbol{\beta} \\ &\quad +2(q-2)k\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta}E[\chi^{-2}_{q+2}(\Delta^2)]. \end{aligned} \tag{4.50}$$

The difference will be non-positive when

$$\boldsymbol{\delta}'[I_p - R(k)'R(k)]\boldsymbol{\delta} \ge \frac{f_4(\Delta^2)}{(q^2-4)E[\chi^{-4}_{q+4}(\Delta^2)]}, \tag{4.51}$$

where

$$
\begin{aligned}
f_4(\Delta^2) &= \sigma_{zz}[tr(R(k)C^{-1}R(k)') - tr(C^{-1})] \\
&\quad +(q-2)\sigma_{zz}[tr(A) - tr(R(k)AR(k)')] \\
&\quad \times \left\{2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right\} \\
&\quad +k^2\boldsymbol{\beta}'C^{-2}(k)\boldsymbol{\beta} + 2(q-2)k\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta}E[\chi_{q+2}^{-2}(\Delta^2)].
\end{aligned}
$$

Since $\Delta^2 > 0$, we assume that the numerator of (4.51) is positive. Then the $\hat{\boldsymbol{\beta}}_n^S$ is superior to $\hat{\boldsymbol{\beta}}_n^S(k)$ when

$$
\Delta^2 \geq \frac{f_4(\Delta^2)}{(q^2-4)Ch_{\max}[(I_p - R(k)'R(k))C^{-1}]E[\chi_{q+4}^{-4}(\Delta^2)]} = \Delta_5^2(k), \quad \text{say}
$$

where $Ch_{\max}(M)$ is the maximum characteristic root of the matrix (M). However, $\hat{\boldsymbol{\beta}}_n^S$ is superior to $\hat{\boldsymbol{\beta}}_n^S(k)$ when

$$
\Delta^2 < \frac{f_4(\Delta^2)}{(q^2-4)Ch_{\min}[(I_p - R(k)'R(k))C^{-1}]E[\chi_{q+4}^{-4}(\Delta^2)]} \ ,
$$

where $Ch_{\min}(M)$ is the minimum characteristic root of the matrix (M).

4.6 Comparison of $\hat{\boldsymbol{\beta}}_n^S(k)$ with $\tilde{\boldsymbol{\beta}}_n(k)$ and $\hat{\boldsymbol{\beta}}_n(k)$:

4.6.1 Comparison of $\hat{\boldsymbol{\beta}}_n^S(k)$ and $\tilde{\boldsymbol{\beta}}_n(k)$:

First we compare $\hat{\boldsymbol{\beta}}_n^S(k)$ and $\tilde{\boldsymbol{\beta}}_n(k)$.

Consider the difference in the risks of $\hat{\boldsymbol{\beta}}_n^S(k)$ and $\tilde{\boldsymbol{\beta}}_n(k)$ as

$$
\begin{aligned}
&R_1(\tilde{\boldsymbol{\beta}}_n(k); I_p) - R_4(\hat{\boldsymbol{\beta}}^S(k); I_p) = \\
&\sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{\left[(q-2)\sigma_{zz}\lambda_i^2 a_{ii}^* \left[(q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right.\right.\right. \\
&+ \left.\left.\left(1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2\sigma_{zz}a_{ii}^*}\right)(2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)]\right]\right] \\
&-2(q-2)k\theta_i\lambda_i\delta_i^* E[\chi_{q+2}^{-2}(\Delta^2)]\Big\}.
\end{aligned}
\tag{4.52}
$$

A sufficient condition for the risk difference to be non-negative whenever $0 < k < k_8(\Delta^2)$ where

$$k_8(\Delta^2) = \frac{f_5(\Delta^2)}{g_3(\Delta^2)}, \tag{4.53}$$

where

$$\begin{aligned} f_5(\Delta^2) &= \min_{1\leq i\leq p} \Big\{\sigma_{zz}a_{ii}^*\lambda_i^2[(q-2)E[\chi_{q+2}^{-4}(\Delta^2)]] \\ &\quad + \left(1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2\sigma_{zz}a_{ii}^*}\right)(2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)]\Big\}, \end{aligned} \tag{4.54}$$

and

$$g_3(\Delta^2) = \max_{1\leq i\leq p}\left[2\theta_i\lambda_i\delta_i^* E[\chi_{q+2}^{-2}(\Delta^2)]\right]. \tag{4.55}$$

Suppose $k > 0$, then the following statements hold true following Kaciranlar, Sakallioglu, Akdeniz, Styan and Werner (1999):

1. If $g_3(\Delta^2) > 0$, it follows that for each $k > 0$ with $k < k_8(\Delta^2)$, $\hat{\boldsymbol{\beta}}_n^S(k)$ has smaller risk than that of $\tilde{\boldsymbol{\beta}}_n(k)$.
2. If $g_3(\Delta^2) < 0$, it follows that for each $k > 0$ with $k > k_8(\Delta^2)$, $\hat{\boldsymbol{\beta}}_n^S(k)$ has smaller risk than that of $\tilde{\boldsymbol{\beta}}_n(k)$.

Note that the risk difference (4.52) under $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$ is

$$\sum_{i=1}^{p} \frac{(q-2)\sigma_{zz}\lambda_i^2 a_{ii}^*}{q(\lambda_i + k)^2} \geq 0.$$

Therefore $\hat{\boldsymbol{\beta}}_n^S(k)$ always dominates $\tilde{\boldsymbol{\beta}}_n(k)$ under the null hypothesis for $q \geq 3$.

4.6.2 Comparison of $\hat{\boldsymbol{\beta}}_n^S(k)$ and $\hat{\boldsymbol{\beta}}_n(k)$:

Now consider the comparison of $\hat{\boldsymbol{\beta}}_n^S(k)$ and $\hat{\boldsymbol{\beta}}_n(k)$. Consider the difference in the risks of $\hat{\boldsymbol{\beta}}_n^S(k)$ and $\hat{\boldsymbol{\beta}}_n(k)$

$$\begin{aligned} & R_4(\hat{\boldsymbol{\beta}}_n^S(k); I_p) - R_2(\hat{\boldsymbol{\beta}}_n(k); I_p) = \\ & \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_{zz} \lambda_i^2 a_{ii}^* \left[1 - (q-2) \left[(q-2) E[\chi_{q+2}^{-4}(\Delta^2)] \right. \right. \right. \\ & \left. \left. + \left(1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2 \sigma_{zz} a_{ii}^*} \right) (2\Delta^2) E[\chi_{q+4}^{-4}(\Delta^2)] \right] \right] \\ & \left. -\lambda_i^2 \delta_i^{*2} - 2k\theta_i \lambda_i \delta_i^* [1 - (q-2) E[\chi_{q+2}^{-2}(\Delta^2)]] \right\}. \end{aligned} \tag{4.56}$$

A sufficient condition for the difference to be non-negative is that $0 < k < k_9(\Delta^2)$ where

$$k_9(\Delta^2) = \frac{f_6(\Delta^2)}{g_4(\Delta^2)}, \tag{4.57}$$

where

$$\begin{aligned} f_6(\Delta^2) &= \max_{1 \le i \le p} \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 \left\{ 1 - (q-2) \left[(q-2) E[\chi_{q+2}^{-4}(\Delta^2)] \right] \right. \right. \\ & \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\Delta^2 \sigma_{zz} a_{ii}^*} \right) (2\Delta^2) E[\chi_{q+4}^{-4}(\Delta^2)] \right\} - \lambda_i^2 \delta_i^{*2} \right\}, \end{aligned} \tag{4.58}$$

with

$$g_4(\Delta^2) = \min_{1 \le i \le p} \left[2\theta_i \lambda_i \delta_i^* [1 - (q-2) E[\chi_{q+2}^{-2}(\Delta^2)]] \right]. \tag{4.59}$$

Suppose $k > 0$, then the following statements hold true:

1. If $g_4(\Delta^2) > 0$, it follows that for each $k > 0$ with $k < k_9(\Delta^2)$, $\hat{\boldsymbol{\beta}}_n^S(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n(k)$.

2. If $g_4(\Delta^2) < 0$, it follows that for each $k > 0$ with $k > k_9(\Delta^2)$, $\hat{\boldsymbol{\beta}}_n(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n^S(k)$.

Note that the risk difference (4.56) under $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$ is

$$\frac{2\sigma_{zz}}{q} \sum_{i=1}^{p} \frac{a_{ii}^*}{(\lambda_i + k)^2} \ge 0.$$

Thus $\hat{\boldsymbol{\beta}}_n(k)$ is superior to $\hat{\boldsymbol{\beta}}_n^S(k)$ under $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$.

Consider the difference in the risks of $\hat{\boldsymbol{\beta}}_n^S(k)$ and $\tilde{\boldsymbol{\beta}}_n^{PT}(k)$ from Theorem 3(iii) and Theorem 3(iv) as

$$\begin{aligned}
&R(\tilde{\boldsymbol{\beta}}_n^S(k); I_p) - R(\hat{\boldsymbol{\beta}}_n^{PT}(k); I_p) \\
&= \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \left\{\sigma_{zz} a_{ii}^* \left[\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right.\right. \\
&- (q-2)\left[(q-2)E[\chi_{q+2}^{-4}(\Delta^2)] + \left(1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2\sigma_{zz}a_{ii}^*}\right)(2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)]\right]\Bigg] \\
&-\lambda_i^2\delta_i^{*2}\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right] \\
&\left.-2k\theta_i\lambda_i\delta_i^*\left[\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2)E[\chi_{q+2}^{-2}(\Delta^2)]\right]\right\}.
\end{aligned} \tag{4.60}$$

We define

$$k_{10}(\alpha, \Delta^2) = \frac{f_8(\alpha, \Delta^2)}{g_5(\alpha, \Delta^2)}, \tag{4.61}$$

where

$$\begin{aligned}
f_8(\alpha, \Delta^2) \;=\; & \max_{1\le i\le p} \left\{\sigma_{zz}a_{ii}^*\lambda_i^2 \left\{\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2)\left[(q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right]\right.\right. \\
& \left. + \left(1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2\sigma_{zz}a_{ii}^*}\right)(2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)]\right\} \\
& \left. -\lambda_i^2\delta_i^{*2}\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right]\right\}.
\end{aligned} \tag{4.62}$$

and

$$g_5(\Delta^2) = \min_{1\le i\le p}\left[2\theta_i\lambda_i\delta_i^{*2}\left[\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right]\right]. \tag{4.63}$$

Suppose $k > 0$, then the following statements hold true:

1. If $g_5(\Delta^2) > 0$, it follows that for each $k > 0$ with $k > k_{10}(\alpha, \Delta^2)$, $\hat{\boldsymbol{\beta}}_n^S(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$.

2. If $g_5(\alpha, \Delta^2) < 0$, it follows that for each $k > 0$ with $k < k_{10}(\alpha, \Delta^2)$, $\hat{\boldsymbol{\beta}}_n^S(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$.

Remark: Now we obtain the conditions for the superiority of $\hat{\boldsymbol{\beta}}_n^S(k)$ over $\hat{\boldsymbol{\beta}}_n(k)$ when $\alpha = 0$ and of $\hat{\boldsymbol{\beta}}_n^S(k)$ over $\tilde{\boldsymbol{\beta}}_n(k)$ when $\alpha = 1$.

The risk difference (4.60) under $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$ reduces to

$$\sum_{i=1}^{p} \frac{\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha);0] - \left(\frac{q-2}{q}\right)\right] \sigma_{zz} a_{ii}^*}{(\lambda_i + k)^2}.$$

Therefore the risk of $\hat{\boldsymbol{\beta}}_n^S(k)$ is smaller than the risk of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ when

$$\chi_q^2(\alpha) \leq \mathcal{H}_{q+2}^{-1}\left[\frac{q-2}{q}, 0\right] \tag{4.64}$$

where $\chi_q^2(\alpha)$ is the upper α-level critical value from the chi-square distribution with q degrees of freedom. Otherwise the risk of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ is smaller than the risk of $\hat{\boldsymbol{\beta}}_n^S(k)$.

4.7 Comparison of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ with $\hat{\boldsymbol{\beta}}_n^{S+}$, $\tilde{\boldsymbol{\beta}}_n(k)$, $\hat{\boldsymbol{\beta}}_n(k)$, $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ and $\hat{\boldsymbol{\beta}}_n^S(k)$:

4.7.1 Comparison of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ and $\hat{\boldsymbol{\beta}}_n^S(k)$:

Case 1: Under the null hypothesis $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$

The risk function of $\hat{\boldsymbol{\beta}}_n^{S+}(k))$ can be expressed as

$$\begin{aligned} R(\hat{\boldsymbol{\beta}}_n^S(k); I_p) &= \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_{zz} \left[\lambda_i - \left(\frac{q-2}{q}\right) a_{ii}^*\right] + k^2\theta_i^2 \right. \\ &\quad \left. -\sigma_{zz} a_{ii}^* E\left[\left(1 - \frac{q-2}{q+2} H(\chi^2(\alpha);0)\right)^2 I\left(H_{q+2}(\chi^2(\alpha);0) < \frac{q-2}{q+2}\right)\right]\right\}. \end{aligned} \tag{4.65}$$

Differentiating (4.65) with respect to k gives

$$\begin{aligned} \frac{\partial R(\hat{\boldsymbol{\beta}}_n^{S+}(k); I_p)}{\partial k} &= 2\sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^3} \left[k\lambda_i\theta_i^2 - \sigma_{zz}\left(\lambda_i - \frac{q-2}{q}\right) a_{ii}^*\lambda_i^2 - a_{ii}^*\lambda_i^2 \right. \\ &\quad \left. \times E\left[1 - \frac{q-2}{q+2} H^{-1}(\chi^2(\alpha);0)\right]^2 I\left(H_{q+2}(\chi^2(\alpha);0) < \frac{q-2}{q+2}\right)\right]. \end{aligned} \tag{4.66}$$

A sufficient condition for (4.66) to be negative is that $k \in (0, k_{11}(\alpha))$ where

$$\begin{aligned} &k_{11}(\alpha) = \\ &\frac{\min_{1\leq i\leq p} \sigma_{zz} \left\{1 - \left(\frac{q-2}{q}\right)\lambda_i a_{ii}^* - \lambda_i a_{ii}^* E\left[1 - \left(\frac{q-2}{q+2}\right) H_{q+2}^{-1}(\chi_q^2(\alpha);0)\right]^2 I\left[H_{q+2}(\chi_q^2(\alpha);0) < \frac{q-2}{q+2}\right]\right\}}{\max_{1\leq i\leq p} \theta_i^2}. \end{aligned} \tag{4.67}$$

Suppose the numerator of (4.67) is positive, then $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ dominates $\hat{\boldsymbol{\beta}}_n^{S+}$ when $k > 0$ belongs to the region $k \in (0, k_{11}(\alpha))$.

Case 2: Under the alternative hypothesis $H_1 : H\boldsymbol{\beta} \neq \boldsymbol{h}$

The risk function of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ can be expressed as

$$\begin{aligned}
&R(\hat{\boldsymbol{\beta}}_n^{S+}(k); I_p) = R(\hat{\boldsymbol{\beta}}_n^{S}(k); I_p) \\
&-\sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \left\{ \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 E\left[\left(1 - \frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) \leq \frac{q-2}{q+2}\right)\right] \right.\right. \\
&\left. +\lambda_i^2 \delta_i^{*2} E\left[\left(1 - \frac{q-2}{q+4}\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^2(\Delta^2) \leq \frac{q-2}{q+4}\right)\right]\right\} \\
&+2\lambda_i^2 \delta_i^{*2} E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) \leq \frac{q-2}{q+2}\right)\right] \\
&\left. +2k\theta_i \lambda_i \delta_i^* E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) \leq \frac{q-2}{q+2}\right)\right]\right\}
\end{aligned} \tag{4.68}$$

where $R(\hat{\boldsymbol{\beta}}_n^{S}(k); I_p)$ is given by (4.46). Differentiating $R(\hat{\boldsymbol{\beta}}_n^{S+}(k); I_p)$ with respect to k, we obtain

Differentiating (4.45) with respect to k gives

$$\begin{aligned}
&\frac{\partial R(\hat{\boldsymbol{\beta}}_n^{S+}(k); I_p)}{\partial k} = \frac{\partial R(\hat{\boldsymbol{\beta}}_n^{S}(k); I_p)}{\partial k} \\
&+2\sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^3} \left\{ \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 E\left[\left(1 - \frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) \leq \frac{q-2}{q+2}\right)\right] \right.\right. \\
&\left. +\lambda_i^2 \delta_i^{*2} E\left[\left(1 - \frac{q-2}{q+4}\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^2(\Delta^2) \leq \frac{q-2}{q+4}\right)\right]\right\} \\
&-(\theta_i - 2\delta_i^*)\lambda_i^2 \delta_i^* E\left[\left(\frac{q-2}{q+4}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) \leq \frac{q-2}{q+4}\right)\right] \\
&\left. +k\theta_i \lambda_i \delta_i^* E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) \leq \frac{q-2}{q+2}\right)\right]\right\}
\end{aligned}$$

where $\partial R(\hat{\boldsymbol{\beta}}_n^{S}(k); I_p)/\partial k$ is given by (4.46). We define

$$k_{12}(\Delta^2) = \frac{f_9(\Delta^2)}{g_6(\Delta^2)}, \tag{4.69}$$

where

$$
\begin{aligned}
f_9(\alpha, \Delta^2) \quad = \quad & \min_{1\le i\le p} \left\{\sigma_{zz} \left\{\lambda_i - (q-2)a^*_{ii}\left[(q-2)E[\chi^{-2}_{q+2}(\Delta^2)]\right]\right.\right. && (4.70)\\
& + \left(1 - \frac{(q+2)\lambda_i\delta_i^{*2}}{2\sigma_{zz}\Delta^2 a^*_{ii}}\right)(2\Delta^2)E[\chi^{-4}_{q+4}(\Delta^2)] \\
& -a^*_{ii}E\left[\left(1-\frac{q-2}{q+2}\chi^{-2}_{q+2}(\Delta^2)\right)^2 I\left(\chi^2_{q+2}(\Delta^2)\le\frac{q-2}{q+2}\right)\right] \\
& -\lambda_i^2\delta_i^{*2}E\left[\left(1-\frac{q-2}{q+4}\chi^{-2}_{q+4}(\Delta^2)\right)^2 I\left(\chi^2_{q+4}(\Delta^2)\le\frac{q-2}{q+4}\right)\right] \\
& +(\theta_i - 2\delta_i^*)\lambda_i^2\delta_i^*E\left[\left(\frac{q-2}{q+2}\chi^{-2}_{q+2}(\Delta^2)-1\right) I\left(\chi^2_{q+2}(\Delta^2)\le\frac{q-2}{q+2}\right)\right] \\
& \left.\left. +dq\theta_i\delta_i^*\lambda_i^2E[\chi^{-2}_{q+2}(\Delta^2)]\right\}\right\} && (4.71)
\end{aligned}
$$

and

$$
\begin{aligned}
g_6(\alpha, \Delta^2) \quad = \quad & \max_{1\le i\le p}\left[\lambda_i\theta_i\left\{\theta_i + \delta_i^*E\left(\frac{q-2}{q+2}\chi^{-2}_{q+2}(\Delta^2)-1\right)I\left(\chi^2_{q+2}(\Delta^2)\le\frac{q-2}{q+2}\right)\right.\right. \\
& \left.\left. -(q-2)\delta_i^*E\left[\chi^{-2}_{q+2}(\Delta^2)\right]\right\}\right]. && (4.72)
\end{aligned}
$$

Suppose $k > 0$, then the following statements hold true following Kaciranlar, Sakallioglu, Akdeniz, Styan and Werner (1999):

1. If $g_6(\Delta^2) > 0$, it follows that for each $k > 0$ with $k < k_{12}(\Delta^2)$, $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n^{S+}$.

2. If $g_6(\Delta^2) < 0$, it follows that for each $k > 0$ with $k > k_{12}(\Delta^2)$, $\hat{\boldsymbol{\beta}}_n^{S+}$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$.

To obtain a condition on Δ^2, we consider the risk difference between $\boldsymbol{\beta}_n^{S+}(k)$ and $\boldsymbol{\beta}_n^{S+}$

as follows:

$$
\begin{aligned}
&R(\hat{\boldsymbol{\beta}}_n^{S+}(k); I_p) - R(\hat{\boldsymbol{\beta}}_n^{S+}; I_p) \qquad (4.73)\\
&= \sigma_{zz}\left[tr(R(k)C^{-1}R(k)') - tr(C^{-1})\right]\\
&+\sigma_{zz}\left[tr(A) - tr(R(k)'AR(k))\right]\left\{2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right\}\\
&+E\left[\left(1 - \frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\\
&+2k\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta}\left\{(q-2) - E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\right\}\\
&-\boldsymbol{\delta}'[I_p - R(k)'R(k)]\boldsymbol{\delta}E^*(\Delta^2)
\end{aligned}
$$

where

$$
\begin{aligned}
E^*(\Delta^2) &= (q-2) - E\left[\left(1 - \frac{q-2}{q+4}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^2(\Delta^2) < \frac{q-2}{q+4}\right)\right]\\
&\quad -2\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right).
\end{aligned}
$$

The right hand side of (4.73) is non-positive when

$$
\boldsymbol{\delta}'[I_p - R(k)'R(k)]\boldsymbol{\delta} \geq \frac{f_{10}(\Delta^2)}{E^*(\Delta^2)} \qquad (4.74)
$$

where

$$
\begin{aligned}
f_{10}(\Delta^2, k) &= \sigma_{zz}\left[tr(R(k)'C^{-1}R(k)) - tr(C^{-1})\right]\\
&\quad +\sigma_{zz}\left[tr(A) - tr(R(k)'AR(k))\right]\left\{2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right\}\\
&\quad +E\left[\left(1 - \frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\\
&\quad +2k\boldsymbol{\delta}'R(k)'C^{-1}(k)\boldsymbol{\beta}\,\{(q-2)\\
&\quad -E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\Big\}.
\end{aligned}
$$

Since $\Delta^2 > 0$, assume that both the numerator and the denominator of (4.74) are positive or negative respectively. Then $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ dominates $\hat{\boldsymbol{\beta}}_n^{S+}$ when

$$
\Delta^2 > \Delta_9^2(k) = \frac{f_{10}(\Delta^2, k)}{Ch_{\max}[I_p - R(k)'R(k)C^{-1}]E^*(\Delta^2)} \qquad (4.75)
$$

and $\hat{\boldsymbol{\beta}}_n^{S+}$ dominates $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ when

$$
\Delta^2 < \Delta_{10}^2(k) = \frac{f_{10}(\Delta^2, k)}{Ch_{\min}[I_p - R(k)'R(k)C^{-1}]E^*(\Delta^2)}. \qquad (4.76)
$$

4.7.2 Comparison of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ with $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ and $\hat{\boldsymbol{\beta}}_n^{S}(k)$:

Since $\tilde{\boldsymbol{\beta}}_n^{S+}(k)$ and $\hat{\boldsymbol{\beta}}_n(k)$ are the particular cases of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$, therefore the comparison between $\tilde{\boldsymbol{\beta}}_n(k)$ and $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ as well as between $\hat{\boldsymbol{\beta}}_n(k)$ and $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ can be skipped.

4.7.3 Comparison between $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ and $\hat{\boldsymbol{\beta}}_n^{PT}(k)$:

Case 1: Under the null hypothesis $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$

The risk difference is

$$\begin{aligned}
& R(\hat{\boldsymbol{\beta}}_n^{S+}(k); I_p) - R(\hat{\boldsymbol{\beta}}_n^{PT}(k); I_p) \\
= & \sum_{i=1}^{p} \frac{\sigma_{zz} a_{ii}^* \lambda_i^2}{(\lambda_i + k)^2} \big\{ \big[\mathcal{H}_{q+2}[\chi_q^2(\alpha); 0] \\
& -(q-2) - E\left[\left(1 - \frac{q-2}{q+2}\mathcal{H}_{q+2}^{-1}\left(\chi_q^2(\alpha), 0\right)\right)^2\right] I\left(\chi_{q+2}^2 \le \frac{q-2}{q+2}\right)\bigg]\bigg\} \ge 0,
\end{aligned} \tag{4.77}$$

for all α satisfying the condition

$$\left\{\alpha : \chi_q^2(\alpha) \ge \mathcal{H}_{q+2}^{-1}\left(q - 2 + E\left[\left(1 - \frac{q-2}{q+2}\chi_{q+2}^{-2}(0)\right)^2 I\left(\mathcal{H}_{q+2}(0) \le \frac{q-2}{q+2}\right)\right]\right)\right\}. \tag{4.78}$$

Thus the risk of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ is smaller than the risk of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ when the critical value $\chi_q^2(\alpha)$ satisfies the condition (4.78). However the risk of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ is smaller than the risk of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ when the critical value $\chi_q^2(\alpha)$ satisfies (4.78) with reverse inequality sign.

Case 2: Under the alternative hypothesis $H_1 : H\boldsymbol{\beta} \ne \boldsymbol{h}$

Under the alternative hypothesis, the difference in the risks of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ and $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ is

$$
\begin{aligned}
&R(\tilde{\boldsymbol{\beta}}_n^{S+}(k); I_p) - R(\hat{\boldsymbol{\beta}}_n^{PT}(k); I_p) \qquad (4.79)\\
&= \sigma_{zz} tr[R(k)AR(k)'] \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2)\left(2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right)\right.\\
&\left. -E\left[\left(1 - \frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\right\}\\
&-\boldsymbol{\delta}' R(k)' R(k)\boldsymbol{\delta} \left\{ 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\right.\\
&-(q-2)\left(2E[\chi_{q+2}^{-2}(\Delta^2)] - 2E[\chi_{q+4}^{-4}(\Delta^2)] + (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right)\\
&+E\left[\left(1 - \frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2 < \frac{q-2}{q+2}\right)\right]\\
&\left. +2E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2 < \frac{q-2}{q+2}\right)\right]\right\}\\
&+2k\boldsymbol{\delta}' R(k)' C(k)' \boldsymbol{\beta} \left\{ (q-2)E[\chi_{q+2}^{-2}(\Delta^2)]\right.\\
&\left. -\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\right\}.
\end{aligned}
$$

The right hand side of (4.79) will be non-positive when

$$
\boldsymbol{\delta}' R(k)' R(k)\boldsymbol{\delta} \geq \frac{f_{11}(\alpha, \Delta^2)}{g_7(\alpha, \Delta^2)}, \qquad (4.80)
$$

where

$$
\begin{aligned}
f_{11}(\alpha, \Delta^2) &= \sigma_{zz} tr[R(k)AR(k)'] \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right. \qquad (4.81)\\
&\quad -(q-2)(2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\\
&\quad \left. -E\left[\left(1 - \frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\right\}\\
&\quad +2k\boldsymbol{\delta}' R(k)' C(k)' \boldsymbol{\beta} \left\{ (q-2)E[\chi_{q+2}^{-2}(\Delta^2)] - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\right.\\
&\quad \left. -E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
g_7(\alpha, \Delta^2) &= 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2] - (q^2-2)E[\chi_{q+4}^{-4}(\Delta^2)] \qquad (4.82)\\
&\quad +E\left[\left(1 - \frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\\
&\quad +2E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\right\}.
\end{aligned}
$$

Since $\Delta^2 > 0$, assume that both the numerator and the denominator of (4.80) are positive or negative respectively. Then $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ dominates $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ when

$$\Delta^2 \geq \Delta_{11}^2(\alpha, k) = \frac{f_{11}(\alpha, \Delta^2)}{Ch_{\max}[R(k)'R(k)C^{-1}]g_7(\alpha, \Delta^2)} \tag{4.83}$$

and $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ dominates $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ when

$$\Delta^2 < \Delta_{12}^2(\alpha, k) = \frac{f_{11}(\Delta^2, k)}{Ch_{\min}[R(k)'R(k)C^{-1}]g_7(\alpha, \Delta^2)}. \tag{4.84}$$

Now consider the difference in the risk functions of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ and $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ as a function of eignevalues as follows:

$$\begin{aligned}
&R(\hat{\boldsymbol{\beta}}_n^{S+}(k); I_p) - R(\hat{\boldsymbol{\beta}}_n^{PT}(k); I_p) \qquad (4.85)\\
&= \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \left\{ \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2) \left[(q-2) E[\chi_{q+2}^{-4}(\Delta^2)] \right.\right.\right.\right.\\
&\left.\left. + \left(1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2 \sigma_{zz} a_{ii}^*}\right) (2\Delta^2) E[\chi_{q+4}^{-4}(\Delta^2)] \right] \right\}\\
&- \lambda_i^2 \delta_i^{*2} \left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2] \right]\\
&- \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 E \left[\left(1 - \frac{q-2}{q+2} \chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) \leq \frac{q-2}{q+2}\right) \right] \right.\\
&\left. + \lambda_i^2 \delta_i^{*2} E \left[\left(1 - \frac{q-2}{q+2} \chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^2(\Delta^2) \leq \frac{q-2}{q+2}\right) \right] \right\}\\
&- 2\lambda_i^2 \delta_i^{*2} E \left[\left(\frac{q-2}{q+4} \chi_{q+4}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+4}^2(\Delta^2) \leq \frac{q-2}{q+4}\right) \right]\\
&- 2k\theta_i \lambda_i \delta_i^* \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2) E[\chi_{q+2}^{-2}(\Delta^2)] \right.\\
&\left. + E \left[\left(\frac{q-2}{q+2} \chi_{q+2}^{-2}(\Delta^2) - 1\right) I\left(\chi_{q+2}^2(\Delta^2) \leq \frac{q-2}{q+2}\right) \right] \right\}.
\end{aligned}$$

Now we define

$$k_{13}(\alpha, \Delta^2) = \frac{f_{12}(\alpha, \Delta^2)}{g_8(\alpha, \Delta^2)}, \tag{4.86}$$

where

$$
\begin{aligned}
f_{12}(\alpha,\Delta^2) \;=\; & \max_{1\le i\le p}\Big\{\sigma_{zz}a_{ii}^*\lambda_i^2\Big\{\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]-(q-2)\left[(q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\right] \qquad (4.87)\\
& +\left(1-\frac{(q+2)\delta_i^{*2}}{2\Delta^2\sigma_{zz}a_{ii}^*}\right)(2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)]\Big\}\\
& -\lambda_i^2\delta_i^{*2}\left[2\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]-\mathcal{H}_{q+4}[\chi_q^2(\alpha);\Delta^2]\right]\\
& -\Big\{\sigma_{zz}a_{ii}^*\lambda_i^2E\left[\left(1-\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2)\le\frac{q-2}{q+2}\right)\right]\\
& +\lambda_i^2\delta_i^{*2}E\left[\left(1-\frac{q-2}{q+4}\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^2(\Delta^2)\le\frac{q-2}{q+4}\right)\right]\Big\}\\
& -2\lambda_i^2\delta_i^{*2}E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)-1\right)I\left(\chi_{q+2}^2(\Delta^2)\le\frac{q-2}{q+2}\right)\right]\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
g_8(\alpha,\Delta^2) \;=\; & \min_{1\le i\le p}\Big[2\lambda_i\theta_i\delta_i^*\Big\{\mathcal{H}_{q+2}[\chi_q^2(\alpha);\Delta^2]-(q-2)E[\chi_{q+2}^{-2}(\Delta^2)] \qquad (4.88)\\
& +E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)-1\right)I\left(\chi_{q+2}^2(\Delta^2)\le\frac{q-2}{q+2}\right)\right]\Big\}\Big].
\end{aligned}
$$

Suppose $k>0$, then the following statements hold true following Kaciranlar, Sakallioglu, Akdeniz, Styan and Werner (1999):

1. If $g_8(\alpha,\Delta^2)>0$, it follows that for each $k>0$ with $k<k_{13}(\alpha,\Delta^2)$, $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$.

2. If $g_8(\alpha,\Delta^2)<0$, it follows that for each $k>0$ with $k>k_{13}(\alpha,\Delta^2)$, $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ has smaller risk than that of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$.

Remark: For $\alpha=0$, we obtain the condition for the superiority of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ over $\hat{\boldsymbol{\beta}}_n(k)$ and for $\alpha=1$, we obtain the superiority condition of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ over $\tilde{\boldsymbol{\beta}}_n(k)$.

4.7.4 Comparison of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ and $\hat{\boldsymbol{\beta}}_n^{S}(k)$:

The risk difference of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ and $\hat{\boldsymbol{\beta}}_n^{S}(k)$ is

$$\begin{aligned}
&R(\hat{\boldsymbol{\beta}}_n^{S+}(k); I_p) - R(\hat{\boldsymbol{\beta}}_n^{S}(k); I_p) \\
&= -\left\{\sigma_{zz} tr[R(k)'AR(k)]E\left[\left(1-\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\right. \\
&\left. +[\boldsymbol{\delta}' R(k)'R(k)\boldsymbol{\delta}]E\left[\left(1-\frac{q-2}{q+4}\chi_{q+4}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+4}^2(\Delta^2) < \frac{q-2}{q+4}\right)\right]\right\} \\
&-2[\boldsymbol{\delta}' R(k)'R(k)\boldsymbol{\delta}]E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)-1\right) I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right] \\
&-2k\boldsymbol{\delta}' R(k)'C^{-1}(k)\boldsymbol{\beta} E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)-1\right) I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right].
\end{aligned} \tag{4.89}$$

Case 1: Suppose $\boldsymbol{\delta}' R(k)'C^{-1}(k)\boldsymbol{\beta} > 0$, then the right hand side of (4.89) is negative, since the expectation of a positive random variable is positive. Thus for all Δ^2 and k,

$$R(\hat{\boldsymbol{\beta}}_n^{S+}(k); I_p) \leq R(\hat{\boldsymbol{\beta}}_n^{S}(k); I_p).$$

Therefore under this condition, the $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ not only confirms the inadmissibility of $\hat{\boldsymbol{\beta}}_n^{S}(k)$ but also provides a simple superior estimator for the ill-conditioned data.

Case 2: Suppose $\boldsymbol{\delta}' R(k)'C^{-1}(k)\boldsymbol{\beta} < 0$, then the right hand side of (4.89) is positive when

$$\boldsymbol{\delta}' R(k)'R(k)\boldsymbol{\delta} \geq \frac{f_{13}(\alpha, \Delta^2)}{g_9(\alpha, \Delta^2)}, \tag{4.90}$$

where

$$\begin{aligned}
f_{13}(\alpha, \Delta^2) &= 2k\boldsymbol{\delta}' R(k)'C^{-1}(k)\boldsymbol{\beta} E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)-1\right) I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right] \\
&\quad -\sigma_{zz} tr[R(k)'AR(k)]E\left[\left(1-\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
g_9(\alpha, \Delta^2) &= \left\{E\left[\left(1-\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)\right)^2 I\left(\chi_{q+2}^2(\Delta^2) < \frac{q-2}{q+2}\right)\right]\right. \\
&\quad \left.-2E\left[\left(\frac{q-2}{q+2}\chi_{q+2}^{-2}(\Delta^2)-1\right) I\left(\chi_{q+2}^2 < \frac{q-2}{q+2}\right)\right]\right\}.
\end{aligned}$$

Since $\Delta^2 > 0$, assume that both the numerator and the denominator of (4.90) are positive or negative respectively. Then $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ dominates $\hat{\boldsymbol{\beta}}_n^{S}(k)$ when

$$\Delta^2 > \Delta_{13}^2(\alpha, k) = \frac{f_{13}(\alpha, \Delta^2)}{Ch_{\max}[R(k)'R(k)C^{-1}]g_9(\alpha, \Delta^2)} \tag{4.91}$$

and $\hat{\boldsymbol{\beta}}_n^{S}(k)$ dominates $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ when

$$\Delta^2 < \Delta_{14}^2(\alpha, k) = \frac{f_{13}(\Delta^2, k)}{Ch_{\min}[R(k)'R(k)C^{-1}]g_9(\alpha, \Delta^2)}. \tag{4.92}$$

Thus, it is observed that the $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ does not uniformly dominates the $\tilde{\boldsymbol{\beta}}_n(k)$, $\hat{\boldsymbol{\beta}}_n(k)$, $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ and $\hat{\boldsymbol{\beta}}_n^{S}(k)$.

5 Summary and Conclusions

In this paper, we have combined the idea of the preliminary test and the Stein-rule estimator with the RR approach to obtain a better estimator for the regression parameters $\boldsymbol{\beta}$ in a multiple measurement error model. Accordingly, we considered five RRR-estimators, namely, $\tilde{\boldsymbol{\beta}}_n(k)$, $\hat{\boldsymbol{\beta}}_n(k)$, $\hat{\boldsymbol{\beta}}_n^{PT}(k)$, $\hat{\boldsymbol{\beta}}_n^{S}(k)$ and $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ for estimating the parameters ($\boldsymbol{\beta}$) when it is suspected that the parameter $\boldsymbol{\beta}$ may belong to a linear subspace defined by $H\boldsymbol{\beta} = \boldsymbol{h}$. The performances of the estimators are compared based on the quadratic risk function under both null and alternative hypotheses. Under the restriction H_0, the $\hat{\boldsymbol{\beta}}_n(k)$ performed the best compared with other estimators, however, it performed the worst even when Δ^2 moves away from its origin. Note under the risk of $\tilde{\boldsymbol{\beta}}_n(k)$ is constant while the risk of $\hat{\boldsymbol{\beta}}_n(k)$ is unbounded as Δ^2 goes to ∞. Also under H_0, the risk of $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ is smaller than the risks of $\hat{\boldsymbol{\beta}}_n^{S}(k)$ and $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ for satisfying (4.78) for $q \geq 3$. Thus, neither $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ nor $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ nor $\hat{\boldsymbol{\beta}}_n^{S}(k)$ dominate each other uniformly. Note that the application of $\hat{\boldsymbol{\beta}}_n^{S+}(k)$ and $\hat{\boldsymbol{\beta}}_n^{S}(k)$ is constrained by the requirement $q \geq 3$, while $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ does not need such a constraint. However, the choice of the level of significance of the test has a dramatic impact on the nature of the risk function for the $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ estimator. Thus when $q \geq 3$, one would use $\hat{\boldsymbol{\beta}}_n^{S+}(k)$; otherwise one uses $\hat{\boldsymbol{\beta}}_n^{PT}(k)$ with some optimum size α.

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