Analysis of Variance and Design of Experiments

Results from Matrix Theory and Random Variables

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Lecture 1

Vectors and Matrices

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We need some basic knowledge to understand the topics in the analysis of variance.

We will be using some results on vectors, matrices and linear models.

Vectors:

A vector *Y* is an ordered *n*-tuple of real numbers. A vector can be expressed as a row vector or a column vector as

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

is a column vector of order *n* x 1

and

$Y' = (y_1, y_2, \ldots, y_n)$ **is a row vector of order 1 x *n*.**

Vectors:

If all y_i = 0 for all i = 1,2,…,n then Y' = (0,0,…,0) is called the null vector.

If $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},\ Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$

then

$X + Y = \begin{pmatrix} x_1 + y_2 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix},$ **and** $kY = \begin{pmatrix} ky_1 \\ ky_2 \\ \vdots \\ ky_n \end{pmatrix}$

Vectors:

$$X + (Y + Z) = (X + Y) + Z$$

$$X'(Y + Z) = X'Y + X'Z$$

$$k(X'Y) = (kX)'Y = X'(kY)$$

$$k(X + Y) = kX + kY$$

$$X'Y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

where *k* is a scalar.

Linear combination:

If $X_1, X_2, \ldots, X_m$ **are *m* vectors and** $k_1, k_2, \ldots, k_m$ **are *m* scalars, then**

$$t = \sum_{i=1}^{m} k_i x_i$$

is called the linear combination of $x_1, x_2, \ldots, x_m$

Linear independence:

If $X_1, X_2, \ldots, X_m$ **are** ***m*** **vectors then they are said to be linearly independent if there exist scalars** $k_1, k_2, \ldots, k_m$ **such that**

$$\sum_{i=1}^{m} k_i X_i = 0 \Rightarrow k_i = 0 \quad \textbf{for all} \quad i = 1, 2, \ldots, m.$$

If there exist $k_1, k_2, \ldots, k_m$ **with at least one** k_i **to be nonzero, such that** $\sum_{i=1}^{m} k_i x_i = 0$ **then** $x_1, x_2, \ldots, x_m$ **are said to be linearly dependent.**

Any set of vectors containing the null vector is linearly dependent.

Linear function:

Let $K = (k_1, k_2, \ldots, k_m)'$ **be a** $m \times 1$ **vector of scalars and** $X = (x_1, x_2, \ldots, x_m)'$ **be a** $m \times 1$ **vector of variables, then** $K'X = \sum_{i=1}^{m} k_i x_i$ **is called a linear function or linear form.**

The vector *K* is called the coefficient vector.

For example, the mean of $x_1, x_2, \ldots, x_m$ **can be expressed as**

$$\bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i = \frac{1}{m}(1, 1, \ldots, 1)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \frac{1}{m} 1_m' X$$

where $1_m'$ **is a** $m \times 1$ **vector of all elements unity.**

Contrast:

The linear function $K'X = \sum_{i=1}^{m} k_i x_i$ **is called a contrast in** $x_1, x_2, \ldots, x_m$ **if** $\sum_{i=1}^{m} k_i = 0.$

For example, the linear functions

$$x_1 - x_2, \qquad 2x_1 - 3x_2 + x_3, \qquad \frac{x_1}{2} - x_2 + \frac{x_3}{2}$$

are contrasts.

- **Contrasts** $x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_j$ **are linearly independent for all** $j = 2, 3, \ldots, m.$
- **Every contrast in** $x_1, x_2, \ldots, x_m$ **can be written as a linear combination of (*m* - 1) contrasts** $x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_m.$

Matrix:

A matrix is a rectangular array of real numbers.

For example,

$$A = \begin{pmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \quad\quad \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{pmatrix}$$

is a matrix of order *m* x *n* with *m* rows and *n* columns.

- **If *m* = *n*, then *A* is called a square matrix.**
- **If** $a_{ij} = 0,\ i \neq j, m = n,$ **then *A* is a diagonal matrix and is denoted as** $A = diag(a_{11}, a_{22}, \dots, a_{mm}).$

Matrix:

If *A* is a *m* x *n* matrix, then the matrix obtained by writing the rows of *A* and columns of *A* as columns of *A* and rows of *A* respectively is called the transpose of a matrix *A* and is denoted as *A'* .

- **If then *A* = *A'* then *A* is a symmetric matrix.**
- **A matrix whose all elements are equal to zero is called a null matrix.**
- **An identity matrix is a square matrix of order *p* whose diagonal elements are unity (ones) and all the off diagonal elements are zero. It is denoted as I_p.**

Matrix:

- **If *A* and *B* are matrices of order** $m \times n$ **then** $(A+B)' = A' + B'.$
- **If *A* and *B* are the matrices of order *m* x *n* and *n* x *p* respectively and *k* is any scalar, then**

$$(AB)' = B'A'$$

$$(kA)B = A(kB) = k(AB) = kAB.$$

- **If the orders of matrices *A* is *m* x *n, B* is *n* x *p* and *C* is *n* x *p* then** $A(B+C) = AB + AC.$
- **If the orders of matrices *A* is *m* x *n, B* is *n* x *p* and *C* is *p* x *q* then** $(AB)C = A(BC).$
- **If *A* is the matrix of order *m* x *n* then** $I_m A = AI_n = A.$

Trace of a matrix:

The trace of *n* x *n* matrix *A*, denoted as *tr*(*A*) or *trace*(*A*) is defined to be the sum of all the diagonal elements of *A*, i.e.

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

- **If *A* is of order *m* x *n* and *B* is of order *n* x *m*, then**

$$tr(AB) = tr(BA).$$

- **If *A* is an *m* x *n* matrix, then** $tr(A'A) = tr(AA') = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2$

  and $tr(A'A) = tr(AA') = 0$ **if and only if *A* = 0.**

- **If *A* is an *n* x *n* matrix then** $tr(A') = trA$

Rank of matrix:

The rank of a matrix *A* of *m* x *n* is the number of linearly independent rows in *A*.

Let *B* be any other matrix of order *n* x *q*.

- **A square matrix of order *m* is called non-singular if it has full rank.**
- $rank(AB) \leq \min(rank(A), rank(B)).$
- $rank(AA') = rank(A'A) = rank(A) = rank(A').$
- ***A* is of full row rank if *rank*(*A*) = *m* < *n*.**
- ***A* is of full column rank if *rank*(*A*) = *n* < *m*.**

Inverse of a matrix:

The inverse of a square matrix *A* of order *m*, is a square matrix of order *m*, denoted as A^{-1}, such that $A^{-1}A = AA^{-1} = I_m$.

The inverse of *A* exists if and only if *A* is non-singular.

- $(A^{-1})^{-1} = A.$
- **If *A* is non-singular, then** $(A')^{-1} = (A^{-1})'.$
- **If *A* and *B* are non-singular matrices of the same order, then their product, if defined, is also nonsingular and** $(AB)^{-1} = B^{-1}A^{-1}.$

Idempotent matrix:

A square matrix *A* is called idempotent if $A^2 = AA = A.$

If *A* is an $n \times n$ **idempotent matrix with** $rank(A) = r \leq n$**. Then**

- **eigenvalues of *A* are 1 or 0.**
- $trace(A) = rank(A) = r.$
- **If *A* and *B* are idempotent and *AB* = *BA*, then *AB* is also idempotent.**
- **If *A* is idempotent then (*I* – *A*) is also idempotent. Then *A*(*I* - *A*) = (*I* - *A*)*A* = 0.**

Quadratic forms:

If *A* is a given matrix of order $m \times n$ and *X* and *Y* are two given vectors of order $m \times 1$ and $n \times 1$ respectively

$$X'AY = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, x_i\, y_j$$

where a_{ij} are the nonstochastic elements of *A*.

If *A* is a square and symmetric matrix, then

$$X'AX = a_{11}x_1^2 + ... + a_{mm}x_m^2 + 2a_{12}x_1x_2 + ... + 2a_{m-1,m}x_{m-1}x_m$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_ix_j$$

is called a quadratic form in *m* variables $x_1, x_2, ..., x_m$ or a quadratic form in *X*.

Quadratic forms:

- **The quadratic form** $X'AX$ **and the matrix *A* of the form is called.**
  - **Positive definite if** $X'AX > 0$ **for all** $x \neq 0$**.**
  - **Positive semidefinite if** $X'AX \geq 0$ **for all** $x \neq 0$ **.**
  - **Negative definite if** $X'AX < 0$ **for all** $x \neq 0$**.**
  - **Negative semidefinite if** $X'AX \leq 0$ **for all** $x \neq 0$ **.**
- **If *P* is any nonsingular matrix and *A* is any positive definite matrix (or positive semi-definite matrix) then** $P'AP$ **is also a positive definite matrix (or positive semi-definite matrix).**

Simultaneous linear equations:

The set of *m* linear equations in *n* unknowns $x_1, x_2, \ldots, x_n$ **and scalars** a_{ij} **and** b_i**,** $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$ **of the form**

$$
\begin{aligned}
& a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{aligned}
$$

can be formulated as *AX = b.*

Here *A* is a real matrix of known scalars of order $m \times n$ **called as a coefficient matrix, *X* is** $n \times 1$ **real vector and *b* is** $m \times 1$ **real vector of known scalars. They are given as follows:**

Simultaneous linear equations:

$$A=\begin{pmatrix} a_{11} & a_{12}\dots a_{1n} \\ a_{21} & a_{22}\dots a_{2n} \\ \vdots & \vdots \quad \ddots \quad \vdots \\ a_{m1} & a_{m2}\dots a_{mn} \end{pmatrix},$$

is an $m \times n$ **real matrix called as coefficient matrix,**

$$X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is an $n \times 1$ **vector of variables and**

$$b=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

, is an $m \times 1$ **real vector.**

Simultaneous linear equations:

- If A is a n x n nonsingular matrix, then $AX = b$ has a unique solution.
- Let $B = [A, b]$ is an augmented matrix. A solution to $AX = b$ exist if and only if $rank(A) = rank(B)$.
- If $AX = b$ is consistent then $AX = b$ has a unique solution if and only if $rank(A) = n$