Analysis of Variance and Design of Experiments

General Linear Hypothesis and Analysis of Variance

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Lecture 11

Multiple Comparison Tests Based on Confidence Intervals and Test of Hypothesis for Variance

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One-way classification with fixed effect linear models of full rank:

Let y_{ij}(j = 1,2,…, n_i) be a random sample from the i^{th} normal population with mean β_i and variance $\sigma^2, i = 1, 2, ..., p$, i.e.,

$$Y_{ij} \sim N(\beta_i, \sigma^2), \; j = 1, 2, ..., n_i; \; i = 1, 2, ..., p.$$

The random samples from different populations are assumed to be independent of each other.

These observations follow the set up of linear model

$$Y = X\beta + \varepsilon$$

The null hypothesis of interest is $H_0 : \beta_1 = \beta_2 = ... = \beta_p = \beta$ (say)

and H_1 : At least one $\beta_i \neq \beta_j (i \neq j)$ where β and σ^2 are unknown.

Case of rejection of H_0

If $F > F_{1-\alpha}(p-1, n-p)$, **then** $H_0 : \beta_1 = \beta_2 = \ldots = \beta_p$ **is rejected.**

This means that at least one β_i **is different from others which is responsible for the rejection.**

So the objective is to investigate and find out such β_i **and divide the population into groups such that the means of populations within the groups are the same.**

Multiple comparison based on confidence intervals:

There is one-to-one relationship between the testing of hypothesis and the confidence interval estimation. So the confidence interval can also be used for such comparisons.

The decision rule is

Reject $H_0 : L = 0$ **against** $H_1 : L \neq 0$ **if** $\left|\hat{L}\right| > t_{df} \sqrt{\widehat{Var}(\hat{L})}.$

If this interval includes *L* = 0 between lower and upper confidence limits, then $H_0 : L = 0$ **is accepted. Our objective is to know if the confidence interval contains zero or not.**

We discuss Tukey's and Scheffe's procedures.

Tukey's procedure for multiple comparisons (*T*-method)

The *T*-method uses the distribution of the studentized range statistic.

(The *S*-method (discussed next) utilizes the *F*-distribution).

The *T*-method can be used to make the simultaneous confidence statements about contrasts $(\beta_i - \beta_j)$ among a set of parameters $\{\beta_1, \beta_2, ..., \beta_p\}$ and an estimate s^2 of error variance if certain restrictions are satisfied.

Tukey's procedure for multiple comparisons (*T*-method)

These restrictions have to be viewed according to the given conditions.

For example, one of the restrictions is that $\hat{\beta}_i\text{'s}$ all have equal variances.

In the setup of one way classification, $\bar{Y}_i$ has its mean $\hat{\beta}_i$ and its variance is $\frac{\sigma^2}{n_i}$. This reduces to a simple condition that all $n_i\text{'s}$ Are the same, i.e., $n_i = n$ for all *i* = 1,2,...,*p* so that all the variances are same.

Another assumption is to assume that $\hat{\beta}_1, \hat{\beta}_2, ..., \hat{\beta}_p$ are statistically independent and the only contrasts considered are the $\frac{p(p-1)}{2}$ differences $\{\beta_i - \beta_j, i \neq j = 1, 2, ..., p\}$.

Tukey's procedure for multiple comparisons (*T*-method)

We make the following assumptions:

(i) The $\hat{\beta}_1, \hat{\beta}_2, ..., \hat{\beta}_p$ **are statistically independent**

(ii) $\hat{\beta}_i \sim N(\beta_i, a^2\sigma^2), i = 1, 2, ..., p, \ a > 0$ **is a known constant.**

(iii) ***s*²** **is an independent estimate of** σ^2 **with** γ **degrees of freedom (Here** $\gamma = n - p$**) , i.e.,** $\frac{\gamma s^2}{\sigma^2} \sim \chi^2(\gamma)$. **and**

(iv) ***s*²** **is statistically independent of** $\hat{\beta}_1, \hat{\beta}_2, ..., \hat{\beta}_p$.

Tukey's procedure for multiple comparisons (*T*-method)

The statement of *T*-method is as follows:

Under the assumptions (i)-(iv), the probability is $(1-\alpha)$ **that the values of contrasts** $L=\sum_{i=1}^{p} C_i\beta_i \quad (\sum_{i=1}^{p} C_i = 0)$ **simultaneously satisfy**

$$\hat{L} - Ts\left(\frac{1}{2}\sum_{i=1}^{p}|C_i|\right) \le L \le \hat{L} + Ts\left(\frac{1}{2}\sum_{i=1}^{p}|C_i|\right)$$

where $\hat{L}=\sum_{i=1}^{p} C_i\hat{\beta}_i,\ \hat{\beta}_i$ **is the maximum likelihood (or least squares) estimate of** β_i**,** $T = aq_{\alpha,p,\gamma}$**, with** $q_{\alpha,p,\gamma}$ **being the upper** $100\alpha\%$ **point of the distribution of Studentized range.**

Tukey's procedure for multiple comparisons (*T*-method)

Note that if *L* is a contrast like $\beta_i - \beta_j (i \neq j)$then $\frac{1}{2}\sum_{i=1}^{p}|C_i| = 1$ and the variance is σ^2 so that and $a = 1$ the interval simplifies to

$$(\hat{\beta}_i - \hat{\beta}_j) - Ts \leq \beta_i - \beta_j \leq (\hat{\beta}_i - \hat{\beta}_j) + Ts$$

where $T = q_{\alpha,p,\gamma}$.

Thus the maximum likelihood (or least squares) estimate $\hat{L} = \hat{\beta}_i - \hat{\beta}_j$ of $L = \beta_i - \beta_j$ is said to be significantly different from zero according to *T*-criterion if the interval $(\hat{\beta}_i - \hat{\beta}_j - Ts,\ \hat{\beta}_i - \hat{\beta}_j + Ts)$ does not cover $\beta_i - \beta_j = 0,$ i.e., if $\left|\hat{\beta}_i - \hat{\beta}_j\right| > Ts$

or more general if $\left|\hat{L}\right| > Ts\left(\frac{1}{2}\sum_{i=1}^{p}|C_i|\right).$

Tukey's procedure for multiple comparisons (*T*-method)

The steps involved in the testing now involve the following steps:

- **Compute** $\hat{L}$ **or** $\left(\hat{\beta}_i - \hat{\beta}_j\right)$.
- **Compute all possible pairwise differences.**
- **Compare all the differences with** $q_{\alpha,p,\gamma} \cdot \frac{s}{\sqrt{n}}\left(\frac{1}{2}\sum_{i=1}^{p}|C_i|\right)$.

If $\left|\hat{L}\right|$ **or** $(\hat{\beta}_i - \hat{\beta}_j) > Ts\left(\frac{1}{2}\sum_{i=1}^{p}|C_i|\right)$

then $\hat{\beta}_i$ **and** $\hat{\beta}_j$ **are significantly different where** $T = \frac{q_{\alpha,p,\gamma}}{\sqrt{n}}$.

Tables for *T* are available.

Tukey's procedure for multiple comparisons (*T*-method)

When sample sizes are not equal, then Tukey-Kramer procedure suggests to compare $|\hat{L}|$ with

$$q_{\alpha,p,\gamma}\; s\sqrt{\frac{1}{2}\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}\;\left(\frac{1}{2}\sum_{i=1}^{p}|C_i|\right)$$

or

$$T\sqrt{\frac{1}{2}\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}\;\left(\frac{1}{2}\sum_{i=1}^{p}|C_i|\right).$$

The Scheffe's method (*S*-method) of multiple comparison

***S*-method generally gives shorter confidence intervals then *T*-method.**

It can be used in a number of situations where *T*-method is not applicable, e.g., when the sample sizes are not equal.

A set *L* of estimable functions $\{\psi\}$ **is called a *p*-dimensional space of estimable functions if there exists *p* linearly independent estimable functions** $(\psi_1, \psi_2, \ldots, \psi_p)$ **such that every** ψ **in *L* is of the form** $\psi = \sum_{i=1}^{p} C_i y_i$ **where** $C_1, C_2, \ldots, C_p$ **are known constants. In other words, *L* is the set of all linear combinations of** $\psi_1, \psi_2, \ldots, \psi_p$.

The Scheffe's method (*S*-method) of multiple comparison

Under the assumption that the parametric space Ω **is** $Y \sim N(X\beta, \sigma^2 I)$ **with** $rank(X) = p, \beta = (\beta_1, ..., \beta_p),$ X **is** $n \times p$ **matrix, consider a *p*-dimensional space *L* of estimable functions generated by a set of *p* linearly independent estimable functions**

$\{\psi_1, \psi_2, ..., \psi_p\}$.

The Scheffe's method (*S*-method) of multiple comparison

For any $\psi \in L,$ **let** $\hat{\psi} = \sum_{i=1}^{n} C_i y_i$ **be its least squares (or maximum likelihood) estimator,**

$$Var(\hat{\psi}) = \sigma^2 \sum_{i=1}^{n} C_i^2$$

$$= \sigma_{\psi}^2 \quad \textbf{(say)}$$

and $\hat{\sigma}_{\psi}^2 = s^2 \sum_{i=1}^{n} C_i^2$ **where** s^2 **is the mean square due to error with (*n* - *p*) degrees of freedom.**

The Scheffe's method (*S*-method) of multiple comparison

The statement of *S*-method is as follows:

Under the parametric space Ω **, the probability is** $(1-\alpha)$ **that simultaneously for all** $\psi \in L,$

$$\hat{\psi} - S\hat{\sigma}_{\hat{\psi}} \le \psi \le \hat{\psi} + S\hat{\sigma}_{\hat{\psi}}$$

where the constant

$$S = \sqrt{pF_{1-\alpha}(p, n-p)}.$$

The Scheffe's method (*S*-method) of multiple comparison

For a given space *L* of estimable functions and confidence coefficient $(1-\alpha)$, the least square (or maximum likelihood) estimate $\hat{\psi}$ of $\psi \in L$ will be said to be significantly different from zero according to *S*-criterion if the confidence interval

$$(\hat{\psi} - S\hat{\sigma}_{\hat{\psi}} \le \psi \le \hat{\psi} + S\hat{\sigma}_{\hat{\psi}})$$

does not cover $\psi = 0,$ i.e., if $|\hat{\psi}| > S\hat{\sigma}_{\hat{\psi}}.$

Comparison of Tukey's and Scheffe's methods:

1. Tukey's method can be used only with equal sample size for all factor level but S-method is applicable whether the sample sizes are equal or not.

2. Although, Tukey's method is applicable for any general contrast, the procedure is more powerful when comparing simple pairwise differences and not making more complex comparisons.

3. If only pairwise comparisons are of interest, and all factor levels have equal sample sizes, Tukey's method gives shorter confidence interval and thus is more powerful.

Comparison of Tukey’s and Scheffe’s methods:

4. In the case of comparisons involving general contrasts, Scheffe’s method tends to give narrower confidence interval and provides a more powerful test.

5. Scheffe’s method is less sensitive to the violations of assumptions of normal distribution and homogeneity of variances.

Comparison of Variances

One of the basic assumptions in the analysis of variance is that the samples are drawn from different normal populations with different means but the same variances.

So before going for analysis of variance, the test of hypothesis about the equality of variance is needed to be done.

We discuss the test of equality of two variances and more than two variances.

Comparison of Variance in One Sample:

Assumptions: Sample is drawn from a normal population .

$$x_1, x_2, \ldots, x_n;\; x_i \sim N(\mu, \sigma^2)$$

Test Statistic: $H_0 : \sigma^2 = \sigma_0^2$ **where** σ_0^2 **is known .**

$$\chi_c^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{\sigma^2}$$

Distribution of above test statistic is Chi Square with (*n* - 1) degree of freedom.

Critical values are obtained from the Chi Square table for given level of significance and d.f.

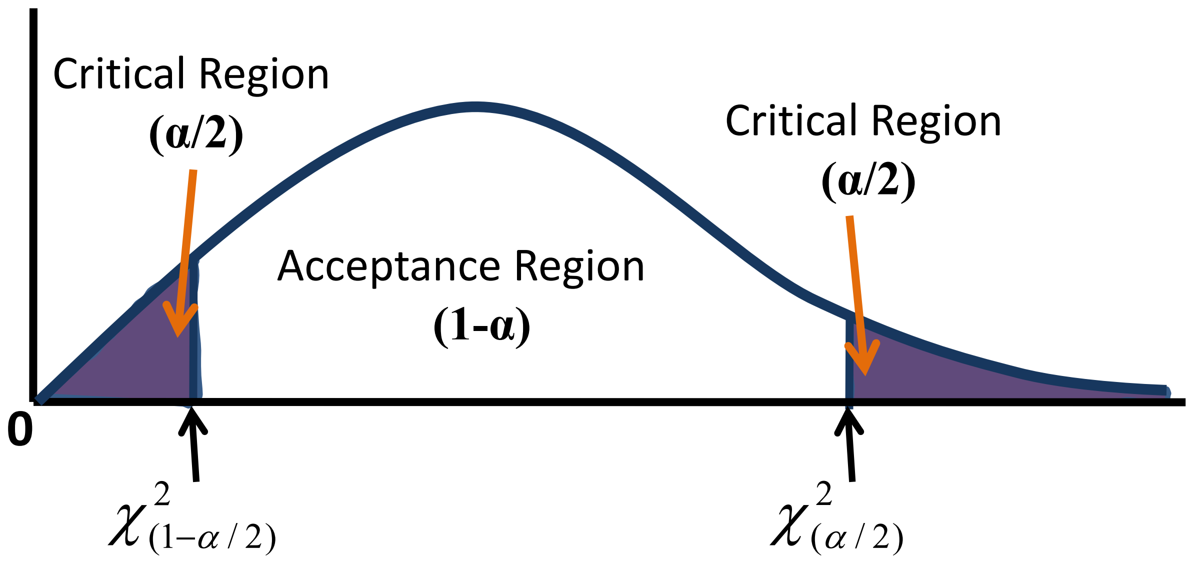
Comparison of Variance in One Sample:

$$H_0 : \sigma^2 = \sigma_0^2$$

$$H_1 : \sigma^2 \neq \sigma_0^2$$

We reject H_0 in the favor of H_1 at α x100% level, if

$$\chi_c^2 > \chi^2_{(\alpha/2)} \quad \textbf{or} \quad \chi_c^2 < \chi^2_{(1-\alpha/2)}$$



Comparison of Variances: Equality of two variances

Suppose there are two independent random samples

$$A: x_1, x_2, \ldots, x_{n_1};\ x_i \sim N(\mu_A, \sigma_A^2)$$

$$B: y_1, y_2, \ldots, y_{n_2};\ y_i \sim N(\mu_B, \sigma_B^2).$$

The sample variance corresponding to the two samples are

$$s_x^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}(x_i - \overline{x})^2, \qquad s_y^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}(y_i - \overline{y})^2.$$

Under $H_0: \sigma_A^2 = \sigma_B^2 = \sigma^2$,

$$\frac{(n_1 - 1)s_x^2}{\sigma^2} \sim \chi^2(n_1 - 1)$$

$$\frac{(n_2 - 1)s_y^2}{\sigma^2} \sim \chi^2(n_2 - 1).$$

Moreover, the sample variances s_x^2 **and** s_y^2 **are independent.**

Comparison of Variances: Equality of two variances

So

$$\left(\frac{\dfrac{\left(\dfrac{(n_1-1)s_x^2}{\sigma^2}\right)}{n_1-1}}{\dfrac{\left(\dfrac{(n_2-1)s_y^2}{\sigma^2}\right)}{n_2-1}} \right) = \frac{s_x^2}{s_y^2} \sim F_{n_1-1,\, n_2-1}.$$

So for testing $H_0: \sigma_A^2 = \sigma_B^2$ **versus** $H_1: \sigma_A^2 \neq \sigma_B^2$ **, the null hypothesis** H_0 **is rejected if** $F > F_{1-\frac{\alpha}{2};n_1-1,n_2-1}$ **or** $F < F_{1-\frac{\alpha}{2};n_1-1,n_2-1}$

where $F_{\frac{\alpha}{2};n_1-1,n_2-1;} = \dfrac{1}{F_{1-\frac{\alpha}{2};n_2-1,n_1-1}}.$

If the null hypothesis $H_0: \sigma_A^2 = \sigma_B^2$ **is rejected, then the problem is termed as the Fisher-Behren's problem. The solutions are available for this problem.**

Comparison of Variances: Equality of more than two variances: Bartlett's test

$H_0 : \sigma_1^2 = \sigma_2^2 = ... = \sigma_k^2$ **and** $H_1 : \sigma_i^2 \neq \sigma_j^2$ **for at least one** $i \neq j = 1, 2, ..., k.$

Let there be *k* independent normal population $N(\mu_i, \sigma_i^2)$ **each of size** $n_i, i = 1, 2, ..., k.$

Let $s_1^2, s_2^2, ..., s_k^2$ **be *k* independent unbiased estimators of population variances** $\sigma_1^2, \sigma_2^2, ..., \sigma_k^2$ **respectively with** $\nu_1, \nu_2, ..., \nu_k$ **degrees of freedom.**

Under $H_0,$ **all the variances are the same as** σ^2 **, say and an unbiased estimate of** σ^2 **is** $s^2 = \sum_{i=1}^{k} \frac{\nu_i s_i^2}{\nu}$ **where** $\nu_i = n_i - 1, \nu = \sum_{i=1}^{k} \nu_i .$

Comparison of Variances: Equality of more than two variances: Bartlett's test

Bartlett has shown that under H_0,

$$\frac{\sum_{i=1}^{k}\left(\nu_i \ln \frac{s^2}{s_i^2}\right)}{\left[1+\frac{1}{3(k-1)}\left\{\sum_{i=1}^{k}\left(\frac{1}{\nu_i}\right)-\frac{1}{\nu}\right\}\right]}$$

is asymptotically distributed as $\chi^2(k-1)$ **based on which** H_0 **can be tested.**