Analysis of Variance and Design of Experiments

Experimental Design Models

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Lecture 14

Two-way classification without interaction in Experimental Design Models

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Two-way classification under fixed effects model:

Suppose the response of an outcome is affected by the two factors – *A* and *B*.

For example, suppose *I* varieties of mangoes are grown on *I* different plots of the same size in each of the *J* different locations. All plots are given the same treatment like an equal amount of water, an equal amount of fertilizer etc.

So there are two factors in the experiment which affect the yield of mangoes.

1. **Location (*A*)**
2. **Variety of mangoes (*B*)**

Such an experiment is called a two – factor experiment.

Two-way classification under fixed effects model:

In this two – factor experiment, different locations correspond to the different levels of *A* and the different varieties correspond to the different levels of factor *B*.

The observations are collected on the basis of per plot.

The combined effect of the two factors (*A* and *B* in our case) is called the interaction effect (of *A* and *B*).

Two-way classification under fixed effects model:

Now there are two options:

- **Only one observation per plot is collected.**
- **More than one observations per plot are collected.**

If there is only one observation per plot then there cannot be any interaction effect among the observations and we assume it to be zero.

If there are more than one observations per plot then the interaction effect among the observations can be considered.

Two-way classification under fixed effects model:

We consider here two cases

1. **One observation per plot in which the interaction effect is zero.**
2. **More than one observations per plot in which the interaction effect is present.**

Two-way classification without interaction: Model

Let y_{ij} be the response of observation from i^{th} level of the first factor, say A and j^{th} level of the second factor, say B.

So assume y_{ij} are independently distributed as

$$N(\mu_{ij}, \sigma^2) \quad i = 1, 2, ..., I, \; j = 1, 2, ..., J.$$

This can be represented in the form of a linear model as

$$\begin{aligned} E(Y_{ij}) &= \mu_{ij} \\ &= \mu_{oo} + (\mu_{io} - \mu_{oo}) + (\mu_{oj} - \mu_{oo}) + (\mu_{ij} - \mu_{io} - \mu_{oj} + \mu_{oo}) \\ &= \mu + \alpha_i + \beta_j + \gamma_{ij} \end{aligned}$$

where $\mu = \mu_{oo}, \quad \alpha_i = \mu_{io} - \mu_{oo}, \quad \beta_j = \mu_{oj} - \mu_{oo}, \quad \gamma_{ij} = \mu_{ij} - \mu_{io} - \mu_{oj} + \mu_{oo}$

with $\sum_{i=1}^{I} \alpha_i = \sum_{i=1}^{I} (\mu_{io} - \mu_{oo}) = 0,$ **and** $\sum_{j=1}^{J} \beta_j = \sum_{j=1}^{J} (\mu_{oj} - \mu_{oo}) = 0.$

Two-way classification without interaction: Model

Here

α_i : **effect of *i*th level of factor *A***

or excess of mean of *i*th level of *A* over the general mean.

β_j : **effect of *j*th level of *B***

or excess of mean of *j*th level of *B* over the general mean.

γ_{ij} : **Interaction effect of *i*th level of *A* and *j*th level of *B*.**

Here we assume $\gamma_{ij} = 0$ **as we have only one observation per plot.**

Two-way classification without interaction: Model

We also assume that the model $E(Y_{ij}) = \mu_{ij}$ is a full rank model so that μ_{ij} and all linear parametric functions of μ_{ij} are estimable.

The total number of observations are *I* x *J* which can be arranged in a two-way classified *I* x *J* table where the rows correspond to the different levels of *A* and the column corresponds to the different levels of *B*.

Two-way classification without interaction: Model

The observations on *Y* and the design matrix *X* in this case are

Y	μ	α_1	α_2	$\cdots$	α_I	β_1	β_2	$\cdots$	β_J
y_{11}	1	1	0	$\cdots$	0	1	0	$\cdots$	0
y_{12}	1	1	0	$\cdots$	0	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots \ddots$		$\vdots$	$\vdots$			
y_{1J}	1	1	0	$\cdots$	0	0	0	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots \ddots$		$\vdots$	$\vdots$	$\vdots \ddots$		$\vdots$
y_{I1}	1	0	0	$\cdots$	1	1	0		0
y_{I2}	1	0	0		1	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots \ddots$		$\vdots$
y_{IJ}	1	0	0		1	0	0		1

Two-way classification without interaction: Hypothesis

If the design matrix is not of full rank, then the model can be reparameterized. In such a case, we can start the analysis by assuming that the model $E(Y_{ij}) = \mu + \alpha_i + \beta_j$ is obtained after reparameterization.

There are two null hypotheses of interest:

$$H_{0\alpha} : \alpha_1 = \alpha_2 = ... = \alpha_I = 0$$

$$H_{0\beta} : \beta_1 = \beta_2 = ... = \beta_J = 0$$

against

$H_{1\alpha}$: **at least one** $\alpha_i (i = 1, 2, ..., I)$ **is different from others**

$H_{1\beta}$: **at least one** $\beta_j (j = 1, 2, ..., J)$ **is different from others.**

Two-way classification without interaction: Estimation

Now we derive the least-squares estimators (or equivalently the maximum likelihood estimator) of μ, α_i **and** $\beta_j,\ i = 1, 2, ..., I,\ j = 1, 2, ..., J$ **by minimizing the error sum of squares**

$$E = \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \mu - \alpha_i - \beta_j)^2.$$

The normal equations are obtained as

$$\frac{\partial E}{\partial \mu} = 0 \Rightarrow -2 \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \mu - \alpha_i - \beta_j) = 0$$

$$\frac{\partial E}{\partial \alpha_i} = 0 \Rightarrow -2 \sum_{j=1}^{J} (y_{ij} - \mu - \alpha_i - \beta_j) = 0\ ,\ i = 1, 2, ..., I$$

$$\frac{\partial E}{\partial \beta_j} = 0 \Rightarrow -2 \sum_{i=1}^{I} (y_{ij} - \mu - \alpha_i - \beta_j) = 0\ ,\ j = 1, 2, ..., J.$$

Solving the normal equations and using $\sum_{i=1}^{I} \alpha_i = 0$ **and** $\sum_{j=1}^{J} \beta_j = 0$**, we get the least-squares estimators of the parameters.**

Two-way classification without interaction: Estimation

Solving the normal equations and using $\sum_{i=1}^{I} \alpha_i = 0$ **and** $\sum_{j=1}^{J} \beta_j = 0$ **, the least-squares estimators are obtained as**

$$\hat{\mu} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} y_{ij} = \frac{G}{IJ} = \overline{y}_{oo}$$

$$\hat{\alpha}_i = \frac{1}{J} \sum_{j=1}^{J} y_{ij} - \overline{y}_{oo} = \frac{T_i}{J} - \overline{y}_{oo} = \overline{y}_{io} - \overline{y}_{oo} \quad i = 1, 2, ..., I$$

$$\hat{\beta}_j = \frac{1}{I} \sum_{i=1}^{I} y_{ij} - \overline{y}_{oo} = \frac{B_j}{I} - \overline{y}_{oo} = \overline{y}_{oj} - \overline{y}_{oo}, \; j = 1, 2, ..., J$$

where

T_i **: treatment totals due to** i^{th} **effect, i.e., the sum of all the observations receiving the** i^{th} **treatment.**

B_j **: block totals due to** j^{th} **effect, i.e., sum of all the observations in the** j^{th} **block.**

Two-way classification without interaction: SSE

Thus the error sum of squares is

$$SSE = \underset{\mu,\alpha_i,\beta_j}{Min}\ E$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j)^2$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}\left[(y_{ij} - \bar{y}_{oo}) - (\bar{y}_{io} - \bar{y}_{oo}) - (\bar{y}_{oj} - \bar{y}_{oo})\right]^2$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{io} - \bar{y}_{oj} + \bar{y}_{oo})^2$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{oo})^2 - J\sum_{i=1}^{I}(\bar{y}_{io} - \bar{y}_{oo})^2 - I\sum_{j=1}^{J}(\bar{y}_{oj} - \bar{y}_{oo})^2$$

which carries $IJ - (I-1) - (J-1) - 1 = (I-1)(J-1)$ **degrees of freedom.**

Two-way classification without interaction: Estimation under $H_{0\alpha}$

Next, we consider the estimation of μ and β_j under the null hypothesis $H_{0\alpha}: \alpha_1 = \alpha_2 = ... = \alpha_I = 0$ by minimizing the error sum of squares $E_1 = \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \mu - \beta_j)^2.$

The normal equations are obtained by

$$\frac{\partial E_1}{\partial \mu} = 0 \text{ and } \frac{\partial E_1}{\partial \beta_j} = 0, \; j = 1, 2, ..., J$$

which on solving gives the least square estimates

$$\hat{\mu} = \bar{y}_{oo}$$

$$\hat{\beta}_j = \bar{y}_{oj} - \bar{y}_{oo}.$$

Two-way classification without interaction: Estimation under $H_{0\alpha}$

The sum of squares due to $H_{0\alpha}$ is

$$\underset{\mu,\beta_j}{Min}\, E_1 = \underset{\mu,\beta_j}{Min} \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \mu - \beta_j)^2$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \hat{\mu} - \hat{\beta}_j)^2$$

$$= J\sum_{i=1}^{I}(\overline{y}_{io} - \overline{y}_{oo})^2 \quad + \quad \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \overline{y}_{io} - \overline{y}_{oj} + \overline{y}_{oo})^2.$$

$\downarrow$ **Sum of squares due to factor A**

$\downarrow$ **Error sum of squares**

Two-way classification without interaction: SSA

Thus the sum of squares due to deviation from $H_{0\alpha}$ (or sum of squares due to rows or sum of squares are to factor *A*)

$$SSA = J\sum_{i=1}^{I}(\overline{y}_{io} - \overline{y}_{oo})^2 = J\sum_{i=1}^{I}\overline{y}_{io}^2 - IJ\overline{y}_{oo}^2$$

and carries $(IJ - J) - (I-1)(J-1) = I - 1$ **degrees of freedom.**

Two-way classification without interaction: Estimation under $H_{0\beta}$

Now we find the estimates of μ and α_i under

$$H_{0\beta} : \beta_1 = \beta_2 = \ldots = \beta_J = 0$$

by minimizing $$E_2 = \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \mu - \alpha_i)^2.$$

The normal equations are

$\frac{\partial E_2}{\partial \mu} = 0$ **and** $\frac{\partial E_2}{\partial \alpha_i} = 0, \; i = 1, 2, \ldots, I$

which on solving give the estimators as

$$\hat{\mu} = \bar{y}_{oo}$$

$$\hat{\alpha}_i = \bar{y}_{io} - \bar{y}_{oo}.$$

Two-way classification without interaction: Estimation under $H_{0\beta}$

The minimum value of the error sum of squares is obtained by

$$\underset{\mu,\alpha_j}{Min}\ E_2 = \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \hat{\mu} - \hat{\alpha}_i)^2$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{io})^2$$

$$= I\sum_{j=1}^{J}(\bar{y}_{oj} - \bar{y}_{oo})^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{io} - \bar{y}_{oj} + \bar{y}_{oo})^2$$

$\downarrow$ **Sum of squares due to factor B**

$\downarrow$ **Error sum of squares**

Two-way classification without interaction: SSB

The sum of squares due to deviation from $H_{0\beta}$ (or the sum of squares due to columns or sum of squares due to factor *B*) is

$$SSB = I\sum_{j=1}^{J}(\overline{y}_{oj} - \overline{y}_{oo})^2 = I\sum_{j}\overline{y}_{oj}^2 - IJ\ \overline{y}_{oo}^2$$

and its degrees of freedom are

$$(IJ - I) - (I - 1)(J - 1) = J - 1.$$

Two-way classification without interaction: Total SS

Note that the total sum of squares is

$$
\begin{aligned}
TSS &= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\overline{y}_{oo})^2 \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J}\left[(\overline{y}_{io}-\overline{y}_{oo})+(\overline{y}_{oj}-\overline{y}_{oo})+(y_{ij}-\overline{y}_{io}-\overline{y}_{oj}+\overline{y}_{oo})\right]^2 \\
&= J\sum_{i=1}^{I}(\overline{y}_{io}-\overline{y}_{oo})^2 + I\sum_{j=1}^{J}(\overline{y}_{oj}-\overline{y}_{oo})^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\overline{y}_{io}-\overline{y}_{oj}+\overline{y}_{oo})^2 \\
&= SSA + SSB + SSE.
\end{aligned}
$$

The partitioning of degrees of freedom into the corresponding groups is $IJ-1=(I-1)+(J-1)+(I-1)(J-1).$

Two-way classification without interaction: Distribution

Note that *SSA, SSB* and *SSE* are mutually orthogonal and that is why the degrees of freedom can be divided like this.

Now using the theory explained while discussing the likelihood ratio test or assuming y_{ij}'s to be independently distributed as

$N(\mu+\alpha_i+\beta_j,\sigma^2),\ i=1,2,...,I;\ j=1,2,...,J,$

and using the Theorems 7 and 9, we can write

$$\frac{SSA}{\sigma^2} \sim \chi^2(I-1)$$

$$\frac{SSB}{\sigma^2} \sim \chi^2(J-1)$$

$$\frac{SSE}{\sigma^2} \sim \chi^2((I-1)(J-1)).$$

Two-way classification without interaction: Distribution

So the test statistic for $H_{0\alpha}$ is obtained as

$$F_1 = \frac{\left(\dfrac{SSA/\sigma^2}{I-1}\right)}{\left(\dfrac{SSE\ /\sigma^2}{(I-1)(J-1)}\right)} = \frac{(I-1)(J-1)}{(I-1)} \cdot \frac{SSA}{SSE}$$

$$= \frac{MSA}{MSE} \sim F((I-1),\ (I-1)\ (J-1)) \text{ under } H_{0\alpha}$$

where $MSA = \dfrac{SSA}{I-1}$ **and** $MSE = \dfrac{SSE}{(I-1)(J-1)}.$

Note that the same statistic is also obtained using the likelihood ratio test for $H_{0\alpha}$.

The decision rule is to reject $H_{0\alpha}$ if $F_1 > F_{1-\alpha}\left[(I-1),(I-1)\ (J-1)\right].$

Under $H_{1\alpha}$, F_1 follows a noncentral *F* distribution

$F(\delta,(J-1),(I-1)(J-1))$ **with** $\delta = \dfrac{J\sum_{i=1}^{I}\alpha_i^2}{\sigma^2}$ **as the non-centrality parameter.**

Two-way classification without interaction: Distribution

Similarly, the test statistic for $H_{0\beta}$ is obtained as

$$F_2 = \frac{\left(\dfrac{SSB/\sigma^2}{J-1}\right)}{\left(\dfrac{SSE/\sigma^2}{(I-1)(J-1)}\right)} = \frac{(I-1)(J-1)}{(J-1)}\frac{SSB}{SSE}$$

$$= \frac{MSB}{MSE} \sim F((J-1),(I-1)(J-1)) \text{ under } H_{0\beta}$$

where $MSB = \dfrac{SSB}{J-1}.$

The decision rule is to reject $H_{0\beta}$ if $F_2 > F_{1-\alpha}((J-1),(I-1)(J-1)).$

The same test statistic can also be obtained from the likelihood ratio test.

Two-way classification without interaction: ANOVA table

The computations of this test of hypothesis can be represented in the form of an analysis of variance table.

ANOVA Table

Source of variation	Degrees of freedom	Sum of squares	Mean squares	*F*-value
Due to Factor *A*	$I - 1$	*SSA*	*MSA*	$F_1 = \frac{MSA}{MSE}$
Due to Factor *B*	$J - 1$	*SSB*	*MSB*	$F_2 = \frac{MSB}{MSE}$
Error	$(I - 1)(J - 1)$	*SSE* (By subtraction)	*MSE*	
Total	$IJ - 1$	*TSS*		

Two-way classification without interaction: Expectations

It can be found on similar lines as in the case of one way classification that

$$E(MSA) = \sigma^2 + \frac{J}{I-1}\sum_{i=1}^{I}\alpha_i^2$$

$$E(MSB) = \sigma^2 + \frac{I}{J-1}\sum_{j=1}^{J}\beta_j^2$$

$$E(MSE) = \sigma^2.$$

Two-way classification without interaction: Multiple comparison

If the null hypothesis is rejected, then we use the multiple comparison tests to divide the α_i's (or β_j's) into groups such that α_i's (or β_j's) belonging to the same group are equal and those belonging to different groups are different.