Analysis of Variance and Design of Experiments

Experimental Design Models

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Lecture 15

Two-way classification with interaction in Experimental Design Models

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Two-way classification with interaction: Model

Consider the two-way classification with an equal number, say *K* observations per cell.

Let y_{ijk} : k^{th} observation in $(I,j)^{th}$ cell , i.e., receiving the treatments

i^{th} level of factor *A* and j^{th} level of factor *B*,

$i = 1, 2, ..., I;\ j = 1, 2, ..., J; k = 1, 2, ..., K$ **and**

y_{ijk} are independently drawn from $N(\mu_{ij}, \sigma^2)$ **so that the linear model under consideration is**

$$y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$$

where ε_{ijk} **are identically and independently distributed following** $N(0, \sigma^2).$

Two-way classification with interaction: Model

Thus $E(y_{ijk}) = \mu_{ij}$

$$= \mu_{oo} + (\mu_{io} - \mu_{oo}) + (\mu_{oj} - \mu_{oo}) + (\mu_{ij} - \mu_{io} - \mu_{oj} + \mu_{oo})$$

$$= \mu + \alpha_i + \beta_j + \gamma_{ij}$$

where $\mu = \mu_{oo}$

$$\alpha_i = \mu_{io} - \mu_{oo}$$

$$\beta_j = \mu_{oj} - \mu_{oo}$$

$$\gamma_{ij} = \mu_{ij} - \mu_{io} - \mu_{oj} + \mu_{oo}$$

with $\sum_{i=1}^{I} \alpha_i = 0, \quad \sum_{j=1}^{J} \beta_j = 0, \quad \sum_{i=1}^{I} \gamma_{ij} = 0, \quad \sum_{j=1}^{J} \gamma_{ij} = 0.$

Assume that the design matrix *X* is of full rank so that all the parametric functions of μ_{ij} **are estimable**

Two-way classification with interaction: Hypothesis

The null and the corresponding alternative hypotheses are

$H_{0\alpha} : \alpha_1 = \alpha_2 = \ldots = \alpha_I = 0$

$H_{1\alpha}$: **At least one** $\alpha_i \neq \alpha_j$, **for** $i \neq j$,

$H_{0\beta} : \beta_1 = \beta_2 = \ldots = \beta_J = 0$

$H_{1\beta}$: **At least one** $\beta_i \neq \beta_j$, **for** $i \neq j$

and

$H_{0\gamma}$: **All** $\gamma_{ij} = 0$ **for all** i, j.

$H_{1\gamma}$: **At least one** $\gamma_{ij} \neq \gamma_{ik}$, **for** $j \neq k$.

Two-way classification with interaction: Estimation

Minimizing the error sum of squares

$$E = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2,$$

The normal equations are obtained as

$$\frac{\partial E}{\partial \mu} = 0,$$

$$\frac{\partial E}{\partial \alpha_i} = 0 \quad \textbf{for all } i,$$

$$\frac{\partial E}{\partial \beta_j} = 0 \quad \textbf{for all } j$$

and

$$\frac{\partial E}{\partial \gamma_{ij}} = 0 \quad \textbf{for all } i \textbf{ and } j$$

Two-way classification with interaction: Least squares estimates

The least-squares estimates are obtained as

$$\hat{\mu} = \bar{y}_{ooo} = \frac{1}{IJK}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} y_{ijk}$$

$$\hat{\alpha}_i = \bar{y}_{ioo} - \bar{y}_{ooo}$$

$$= \frac{1}{JK}\sum_{i=1}^{I} y_{ijk} - \bar{y}_{ooo}$$

$$\hat{\beta}_j = \bar{y}_{ojo} - \bar{y}_{ooo}$$

$$= \frac{1}{IK}\sum_{j=1}^{J} y_{ijk} - \bar{y}_{ooo}$$

$$\hat{\gamma}_{ij} = \bar{y}_{ijo} - \bar{y}_{ioo} - \bar{y}_{ojo} + \bar{y}_{ooo}$$

$$= \frac{1}{K}\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ijk} - \bar{y}_{ioo} - \bar{y}_{ojo} + \bar{y}_{ooo}.$$

Two-way classification with interaction: SSE

The error sum of squares is

$$SSE = \underset{\hat{\mu},\, \hat{\alpha}_i,\, \hat{\beta}_j,\, \hat{\gamma}_{ij}}{Min} \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\gamma}_{ij})^2$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ijo})^2$$

with

$$\frac{SSE}{\sigma^2} \sim \chi^2(IJ(K-1)).$$

Two-way classification with interaction: Estimation under $H_{0\alpha}$

Now minimizing the error sum of squares under

$$H_{0\alpha} = \alpha_1 = \alpha_2 = ... = \alpha_I = 0$$

i.e., minimizing $E_1 = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \mu - \beta_j - \gamma_{ij})^2$

with respect to μ, β_j **and** γ_{ij} **and solving the normal equations**

$$\frac{\partial E_1}{\partial \mu} = 0, \quad \frac{\partial E_1}{\partial \beta_j} = 0 \quad \text{for all } j \quad \text{and} \quad \frac{\partial E_1}{\partial \gamma_{ij}} = 0 \quad \text{for all } i \text{ and } j,$$

gives the least squares estimates as

$$\hat{\mu} = \bar{y}_{ooo}$$

$$\hat{\beta}_j = \bar{y}_{ojo} - \bar{y}_{ooo}$$

$$\hat{\gamma}_{ij} = \bar{y}_{ijo} - \bar{y}_{ooo} - \bar{y}_{ojo} + \bar{y}_{ooo} = \bar{y}_{ijo} - \bar{y}_{ojo}.$$

Two-way classification with interaction: Estimation under $H_{0\alpha}$

Solving the normal equations, we get the least squares estimates as

$$\hat{\mu} = \hat{y}_{ooo}$$

$$\hat{\beta}_j = \overline{y}_{ojo} - \overline{y}_{ooo}$$

$$\hat{\gamma}_{ij} = \overline{y}_{ijo} - \overline{y}_{ojo}.$$

The sum of squares due to $H_{0\alpha}$ **is**

$$\underset{\mu,\beta_j,\gamma_{ij}}{Min} \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \mu - \beta_j - \gamma_{ij})^2$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \hat{\mu} - \hat{\beta}_j - \hat{\gamma}_{ij})^2$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \overline{y}_{ijo})^2 + JK\sum_{i=1}^{I}(\overline{y}_{ioo} - \overline{y}_{ooo})^2$$

$$= SSE + JK \sum_{i=1}^{I}(\overline{y}_{ioo} - \overline{y}_{ooo})^2.$$

Two-way classification with interaction: SSA

Thus the sum of squares due to deviation from $H_{0\alpha}$ or the sum of squares due to effect *A* is

$$SSA = \textbf{Sum of squares due to } H_{0\alpha} - SSE$$

$$= JK\sum_{i=1}^{I}(\bar{y}_{ioo} - \bar{y}_{ooo})^2$$

with

$$\frac{SSA}{\sigma^2} \sim \chi^2(I-1).$$

Two-way classification with interaction: Estimation under $H_{0\beta}$

Minimizing the error sum of squares under $H_{0\beta}: \beta_1 = \beta_2 = ... = \beta_J = 0$**,**

i.e., minimizing $E_2 = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \mu - \alpha_i - \gamma_{ij})^2,$

and solve the normal equations

$$\frac{\partial E_2}{\partial \mu} = 0,$$

$$\frac{\partial E_2}{\partial \alpha_i} = 0 \text{ for all } j$$

and

$$\frac{\partial E_2}{\partial \gamma_{ij}} = 0 \text{ for all } i \text{ and } j.$$

Two-way classification with interaction: Estimation under $H_{0\beta}$

This yields the least-squares estimators as

$$\hat{\mu} = \overline{y}_{ooo}$$

$$\hat{\alpha}_i = \overline{y}_{ioo} - \overline{y}_{ooo}$$

$$\hat{\gamma}_{ij} = \overline{y}_{ijo} - \overline{y}_{ioo}.$$

Two-way classification with interaction: SSB

The minimum error sum of squares is

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-\hat{\mu}-\hat{\alpha}_i-\hat{\gamma}_{ij})^2 = SSE + IK\sum_{j=1}^{J}(\bar{y}_{ojo}-\bar{y}_{ooo})^2$$

and the sum of squares due to deviation from $H_{0\beta}$ **or the sum of squares due to effect *B* is**

$$SSB = \text{Sum of squares due to } H_{0\beta} - SSE$$

$$= IK\sum_{j=1}^{J}(\bar{y}_{ojo}-\bar{y}_{ooo})^2$$

with

$$\frac{SSB}{\sigma^2} \sim \chi^2(J-1).$$

Two-way classification with interaction: Estimation under $H_{0\gamma}$

Next, minimizing the error sum of squares under $H_{0\gamma} : all\ \gamma_{ij} = 0$

for all *i, j*, i.e., minimizing

$$E_3 = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \mu - \alpha_i - \beta_j)^2$$

with respect to μ, α **and** β_j **and solving the normal equations**

$$\frac{\partial E_3}{\partial \mu} = 0, \quad \frac{\partial E_3}{\partial \alpha_i} = 0 \text{ for all } i \text{ and } \frac{\partial E_3}{\partial \beta_j} = 0 \text{ for all } j$$

yields the least-squares estimators

$$\hat{\mu} = \bar{y}_{ooo}$$

$$\hat{\alpha}_i = \bar{y}_{ioo} - \bar{y}_{ooo}$$

$$\hat{\beta}_j = \bar{y}_{ojo} - \bar{y}_{ooo}.$$

Two-way classification with interaction: Estimation under $H_{0\gamma}$

The sum of squares due to $H_{0\gamma}$ **is**

$$\underset{\mu,\alpha_i,\beta_j}{Min}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-\mu-\alpha_i-\beta_j)^2=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-\hat{\mu}-\hat{\alpha}_i-\hat{\beta}_j)^2$$

$$=SSE+K\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ijo}-\bar{y}_{ioo}-\bar{y}_{ojo}+\bar{y}_{ooo})^2.$$

Thus the sum of squares due to deviation from $H_{0\gamma}$ **or the sum of squares due to the interaction effect** ***AB*** **is**

$SSAB =$ **Sum of squares due to** $H_{0\gamma}-SSE$

$$=K\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ijo}-\bar{y}_{ioo}-\bar{y}_{ojo}+\bar{y}_{ooo})^2$$

with

$$\frac{SSAB}{\sigma^2}\sim\chi^2((I-1)J-1)).$$

Two-way classification with interaction: Distribution

The total sum of squares can be partitioned as

$$TSS = SSA + SSB + SSAB + SSE$$

where *SSA, SSB, SSAB* and *SSE* are mutually orthogonal.

So either using the independence of *SSA, SSB, SSAB* and *SSE* as well as their respective χ^2 –distributions or using the likelihood ratio test approach using the Theorems 7 and 9, the decision rules for the null hypothesis at α level of significance are based on *F* - statistics as follows:

Two-way classification with interaction: Decision rule

$$F_1 = \frac{IJ(K-1)}{I-1} . \frac{SSA}{SSE} \sim F\left[(I-1), IJ(K-1)\right] \text{ under } H_{0\alpha},$$

$$F_2 = \frac{IJ(K-1)}{J-1} . \frac{SSB}{SSE} \sim F\left[(J-1), IJ(K-1)\right] \text{ under } H_{0\beta},$$

and

$$F_3 = \frac{IJ(K-1)}{(I-1)(J-1)} . \frac{SSAB}{SSE} \sim F\left[(I-1)(J-1), IJ(K-1)\right] \text{ under } H_{0\gamma}.$$

Reject $H_{0\alpha}$ **if** $F_1 > F_{1-\alpha}\left[(I-1), IJ(K-1)\right]$

Reject $H_{0\beta}$ **if** $F_2 > F_{1-\alpha}\left[(J-1), IJ(K-1)\right]$

Reject $H_{0\gamma}$ **if** $F_3 > F_{1-\alpha}\left[(I-1)(J-1), IJ(K-1)\right].$

Two-way classification with interaction: Decision rule

If $H_{0\alpha}$ or $H_{0\beta}$ is rejected, one can use *t*-test or multiple comparison test to find which pairs of α_i 's or β_j 's are significantly different.

If $H_{0\gamma}$ is rejected, one would not usually explore it further but theoretically, *t*-test or multiple comparison tests can be used.

Two-way classification with interaction: ANOVA

The computations of this test of hypothesis can be represented in the form of an analysis of variance table.

ANOVA Table

Source of variation	Degrees of freedom	Sum of squares	Mean squares	*F*-value
Due to Factor *A*	$I-1$	SSA	MSA	$F_1 = \frac{MSA}{MSE}$
Due to Factor *B*	$J-1$	SSB	MSB	$F_2 = \frac{MSB}{MSE}$
Due to Interaction *AB*	$(I-1)(J-1)$	$SSAB$	$MSAB$	$F_3 = \frac{MSAB}{MSE}$
Error	$IJ(K-1)$	SSE (By subtraction)	MSE	
Total	$IJK-1$	TSS		

Two-way classification with interaction: Expectations of SS

It can also be shown that

$$E(SSA) = \sigma^2 + \frac{JK}{I-1}\sum_{i=1}^{I}\alpha_i^2$$

$$E(SSB) = \sigma^2 + \frac{IK}{J-1}\sum_{j=1}^{J}\beta_j^2$$

$$E(SSAB) = \sigma^2 + \frac{K}{(I-1)(J-1)}\sum_{i=1}^{I}\sum_{j=1}^{J}\gamma_{ij}^2$$

$$E(SSE) = \sigma^2.$$