Analysis of Variance and Design of Experiments

Experimental Design Models

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Lecture 16

Tukey's Test for Non-additivity

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Tukey's test for non-additivity:

Consider the set up of two way classification with one observation per cell and interaction as

$$y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ij}, \quad i = 1, 2..., I, \; j = 1, 2, ..., J$$

with $\sum_{i=1}^{I} \alpha_i = 0, \sum_{j=1}^{J} \beta_j = 0.$

The distribution of degrees of freedom in this case is as follows:

Tukey's test for non-additivity:

Source	Degrees of freedom
A	$I - 1$
B	$J - 1$
AB(interaction)	$(I - 1)(J - 1)$
Error	0
Total	$(IJ - 1)$

There is no degree of freedom for error.

Tukey's test for non-additivity:

The problem is that the two factor interaction effect and random error component are subsumed together and cannot be separated out.

There is no estimate for σ^2.

If no interaction exists, then $H_0 : \gamma_{ij} = 0$ **for all *i, j* is accepted and the additive model**

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

is well enough to test the hypothesis $H_0 : \alpha_i = 0$ **and** $H_0 : \beta_j = 0$ **with error having** $(I-1)(J-1)$ **degrees of freedom.**

Tukey's test for non-additivity:

If interaction exists, then $H_0 : \gamma_{ij} = 0$ is rejected.

In such a case, if we assume that the structure of the interaction effect is such that it is proportional to the product of individual effects, i.e., $\gamma_{ij} = \lambda \alpha_i \beta_j$ then a test for testing $H_0 : \lambda = 0$ can be constructed.

Such a test will serve as a test for non-additivity.

It will help in knowing the effect of the presence of interact effect and whether the interaction enters into the model additively.

Tukey's test for non-additivity:

Such a test is given by Tukey's test for non-additivity which requires one degree of freedom leaving $(I-1)(J-1)-1$ degrees of freedom for error.

Let us assume that departure from additivity can be specified by introducing a product term and writing the model as

$$E(y_{ij}) = \mu + \alpha_i + \beta_j + \lambda\alpha_i\beta_j;\; i = 1,2,...,I,\; j = 1,2,...,J$$

with $\sum_{i=1}^{I}\alpha_i = 0, \sum_{j=1}^{J}\beta_j = 0.$

When $\lambda \neq 0,$ **the model becomes a nonlinear model and the least-squares theory for linear models is not applicable.**

Tukey's test for non-additivity:

Note that using , we have

$$\bar{y}_{oo} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[\mu + \alpha_i + \beta_j + \lambda\alpha_i\beta_j + \varepsilon_{ij}\right]$$

$$= \mu + \frac{1}{I}\sum_{i=1}^{I}\alpha_i + \frac{1}{J}\sum_{j=1}^{J}\beta_j + \frac{\lambda}{IJ}(\sum_{i=1}^{I}\alpha_{j})(\sum_{j=1}^{J}\beta_j) + \bar{\varepsilon}_{oo}$$

$$= \mu + \bar{\varepsilon}_{oo}$$

$$E(\bar{y}_{oo}) = \mu$$

$$\Rightarrow \hat{\mu} = \bar{y}_{oo}.$$

Tukey's test for non-additivity:

Next

$$\bar{y}_{io} = \frac{1}{J}\sum_{j=1}^{J} y_{ij} = \frac{1}{J}\sum_{j=1}^{J}\left[\mu + \alpha_i + \beta_j + \lambda\alpha_i\beta_j + \varepsilon_{ij}\right]$$

$$= \mu + \alpha_i + \frac{1}{J}\sum_{j=1}^{J}\beta_j + \lambda\alpha_i\frac{1}{J}\sum_{j=1}^{J}\beta_j + \bar{\varepsilon}_{io}$$

$$= \mu + \alpha_i + \bar{\varepsilon}_{io}$$

$$E(\bar{y}_{io}) = \mu + \alpha_i$$

$$\Rightarrow \hat{\alpha}_i = \bar{y}_{io} - \hat{\mu} = \bar{y}_{io} - \bar{y}_{oo}.$$

Similarly

$$\bar{y}_{oj} = \mu + \beta_j$$

$$\Rightarrow \hat{\beta}_j = \bar{y}_{oj} - \hat{\mu} = \bar{y}_{oj} - \bar{y}_{oo}.$$

Tukey's test for non-additivity:

Thus $\hat{\mu},\ \hat{\alpha}_i$ **and** $\hat{\beta}_j$ **remain the unbiased estimators of** μ, α_i **and** β_j, **respectively irrespective of whether** $\lambda = 0$ **or not.**

Also

$$E\left[y_{ij} - \overline{y}_{io} - \overline{y}_{oj} + \overline{y}_{oo} \right] = \lambda \alpha_i \beta_j$$

or

$$E\left[(y_{ij} - \overline{y}_{oo}) - (\overline{y}_{io} - \overline{y}_{oo}) - (\overline{y}_{oj} - \overline{y}_{oo}) \right] = \lambda \alpha_i \beta_j .$$

Tukey's test for non-additivity:

Consider the estimation of μ, α_i, β_j and λ based on the minimization of

$$S = \sum_i \sum_j (y_{ij} - \mu - \alpha_i - \beta_j - \lambda \alpha_i \beta_j)^2$$
$$= \sum_i \sum_j S_{ij}^2.$$

Use least squares and observe the outcome.

Tukey's test for nonadditivity:

The normal equations are solved as

$$\frac{\partial S}{\partial \mu} = 0 \Rightarrow \sum_{i=1}^{I}\sum_{j=1}^{J} S_{ij} = 0 \;\Rightarrow \hat{\mu} = \bar{y}_{oo}$$

$$\frac{\partial S}{\partial \alpha_i} = 0 \Rightarrow \sum_{j=1}^{J}(1+\lambda\beta_j)S_{ij} = 0$$

$$\frac{\partial S}{\partial \beta_j} = 0 \Rightarrow \sum_{i=1}^{I}(1+\lambda\alpha_i)S_{ij} = 0$$

$$\frac{\partial S}{\partial \lambda} = 0 \Rightarrow \sum_{i=1}^{I}\sum_{j=1}^{J}\alpha_i\beta_j S_{ij} = 0$$

or $\sum_{i=1}^{I}\sum_{j=1}^{J}\alpha_i\beta_j(y_{ij}-\mu-\alpha_i-\beta_j-\lambda\alpha_i\beta_j) = 0$

or $\lambda = \dfrac{\sum_{i=1}^{I}\sum_{j=1}^{J}\alpha_i\beta_j y_{ij}}{\left(\sum_{i=1}^{I}\alpha_i^2\right)\left(\sum_{j=1}^{J}\beta_j^2\right)} = \tilde{\tilde{\lambda}}$ **(say)** **which can be estimated provided** α_i **and** β_j **are assumed to be known.**

Tukey's test for non-additivity:

Since α_i **and** β_j **can be estimated by** $\hat{\alpha}_i = \bar{y}_{io} - \bar{y}_{oo}$ **and** $\hat{\beta}_j = \bar{y}_{oj} - \bar{y}_{oo}$ **irrespective of whether** $\lambda \neq 0$ **or not, so we can substitute them in place of** α_i **and** β_j **in** $\tilde{\lambda}$ **which gives**

$$\hat{\lambda} = \frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\hat{\alpha}_i\hat{\beta}_j y_{ij}}{\left(\sum_{i=1}^{I}\hat{\alpha}_i^2\right)\left(\sum_{j=1}^{J}\hat{\beta}_j^2\right)} = \frac{(IJ)\sum_{i=1}^{I}\sum_{j=1}^{J}\hat{\alpha}_i\hat{\beta}_j y_{ij}}{\left(J\sum_{i=1}^{I}\hat{\alpha}_i^2\right)\left(I\sum_{j=1}^{J}\hat{\beta}_j^2\right)}$$

$$= \frac{IJ\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{io} - \bar{y}_{oo})(\bar{y}_{oj} - \bar{y}_{oo})y_{ij}}{S_A S_B}$$

where $S_A = J\sum_{i=1}^{I}\hat{\alpha}_i^2 = J\sum_{i=1}^{I}(\bar{y}_{io} - \bar{y}_{oo})^2$

$$S_B = I\sum_{j=1}^{J}\hat{\beta}_j^2 = I\sum_{j=1}^{J}(\bar{y}_{oj} - \bar{y}_{oo})^2.$$

Tukey's test for non-additivity:

Assuming α_i **and** β_j **to be known**

$$Var(\tilde{\tilde{\lambda}}) = \left(\frac{1}{\sum_{i=1}^{I} \alpha_i^2 \sum_{j=1}^{J} \beta_j^2} \right)^2 \left[\sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_i^2 \beta_j^2 Var(y_{ij}) + 0 \right]$$

$$= \frac{\sigma^2 \left(\sum_{i=1}^{I} \alpha_i^2 \right) \left(\sum_{j=1}^{J} \beta_j^2 \right)}{\left(\sum_{i=1}^{I} \alpha_i^2 \right)^2 \left(\sum_{j=1}^{J} \beta_j^2 \right)^2} \quad = \frac{\sigma^2}{\left(\sum_{i=1}^{I} \alpha_i^2 \right) \left(\sum_{j=1}^{J} \beta_j^2 \right)}$$

using

$Var(y_{ij}) = \sigma^2, \qquad Cov(y_{ij}, y_{jk}) = 0$ **for all** $i \neq k.$

Tukey's test for non-additivity:

When α_i and β_j are estimated by $\hat{\alpha}_i$ and $\hat{\beta}_j$ then substitute them back in the expression of $Var(\tilde{\tilde{\hat{\lambda}}})$ and treating it as $Var(\hat{\lambda})$ gives

$$Var(\hat{\lambda}) = \frac{\sigma^2}{\left(\sum_{i=1}^{I} \hat{\alpha}_i^2\right)\left(\sum_{j=1}^{J} \hat{\beta}_j^2\right)}$$

$$= \frac{IJ\sigma^2}{S_A S_B}$$

for given $\hat{\alpha}_i$ and $\hat{\beta}_j$.

Tukey's test for non-additivity:

Note that if $\lambda = 0,$ then

$$E\left[\hat{\lambda} / \hat{\alpha}_i, \hat{\beta}_j \text{ for all } i, j\right] = E\left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\alpha_i\beta_j y_{ij}}{\sum_{i=1}^{I}\alpha_i^2\sum_{j=1}^{J}\beta_j^2}\right]$$

$$= E\left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\alpha_i\beta_j(\mu+\alpha_i+\beta_j+0+\varepsilon_{ij})}{(\sum_{i=1}^{I}\alpha_i^2)(\sum_{j=1}^{J}\beta_j^2)}\right] = \frac{0}{(\sum_{i=1}^{I}\alpha_i^2)\,(\sum_{j=1}^{J}\beta_j^2)} = 0.$$

As $\hat{\alpha}_i$ and $\hat{\beta}_j$ remains valid irrespective of $\lambda = 0,$ or not, in this sense $\hat{\lambda}$ is a function of y_{ij} and hence normally distributed as

$$\hat{\lambda} \sim N\left(0, \ \frac{IJ\sigma^2}{S_A S_B}\right).$$

Tukey's test for non-additivity:

Thus the statistic

$$\frac{(\hat{\lambda})^2}{Var(\hat{\lambda})} = \frac{IJ\left[\sum_{i=1}^{I}\sum_{j=1}^{J}\hat{\alpha}_i\hat{\beta}_j y_{ij}\right]^2}{\sigma^2 S_A S_B} = \frac{IJ\left[\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{io}-\bar{y}_{oo})(\bar{y}_{oj}-\bar{y}_{oo})y_{ij}\right]^2}{\sigma^2 S_A S_B}$$

$$= \frac{IJ\left[\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{io}-\bar{y}_{oo})(\bar{y}_{oj}-\bar{y}_{oo})(y_{ij}-\bar{y}_{io}-\bar{y}_{oj}+\bar{y}_{oo})\right]^2}{\sigma^2 S_A S_B} = \frac{S_N}{\sigma^2}$$

follows a χ^2 - distribution with one degree of freedom where

$$S_N = \frac{IJ\left[\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{io}-\bar{y}_{oo})(\bar{y}_{oj}-\bar{y}_{oo})(y_{ij}-\bar{y}_{io}-\bar{y}_{oj}+\bar{y}_{oo})\right]^2}{S_A S_B}$$

is the sum of squares due to non-additivity.

Tukey's test for non-additivity:

Note that

$$\frac{S_{AB}}{\sigma^2} = \frac{\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \overline{y}_{io} - \overline{y}_{oj} + \overline{y}_{oo})^2}{\sigma^2}$$

Follows $\chi^2((I-1)(J-1))$, **so** $\left(\frac{S_N}{\sigma^2} - \frac{SAB}{\sigma^2}\right)$ **is nonnegative and follows** $\chi^2\left[(I-1)(J-1)-1\right]$

The reason for this is as follows:

$$y_{ij} = \mu + \alpha_i + \beta_j + \textbf{non additivity} + \varepsilon_{ij}$$

and so

$$TSS = SSA + SSB + S_N + SSE$$

$$\Rightarrow SSE = TSS - SSA - SSB - S_N$$

has degrees of freedom

$$(IJ-1) - (I-1) - (J-1) - 1 = (I-1)(J-1) - 1$$

Tukey's test for non-additivity:

We need to ensure that *SSE* > 0. So using the result

"If Q, Q_1 and Q_2 are quadratic forms such that

$Q = Q_1 + Q_2$ with $Q \sim \chi^2(a)$, $Q_2 \sim \chi^2(b)$ and Q_2 is non-negative, then $Q_1 \sim \chi^2(a-b)$"

ensures that the difference $\frac{S_N}{\sigma^2} - \frac{SAB}{\sigma^2}$ is nonnegative.

Moreover S_N (*SS* due to non-additivity) and *SSE* are orthogonal.

Tukey's test for non-additivity:

Thus the *F*-test for non-additivity is

$$F = \frac{\left(\frac{S_N / \sigma^2}{1}\right)}{\left(\frac{SSE / \sigma^2}{(I-1)(J-1)-1}\right)}$$

$$= \left[(I-1)(J-1)-1\right]\frac{SSN}{SSE}$$

$\sim F\left[1,(I-1)(J-1)-1\right]$ **under** H_0.

So the decision rule is

Reject $H_0 : \lambda = 0$ **whenever**

$$F > F_{1-\alpha}\left[1,(I-1)(J-1)-1\right].$$

Tukey's test for non-additivity:

The analysis of variance table for the model including a term for non-additivity is as follows:

ANOVA Table

Source of variation	Degrees of freedom	Sum of squares	Mean squares	*F*-value
Due to Factor *A*	$I-1$	S_A	MSA	
Due to Factor *B*	$J-1$	S_B	MSB	
Due to Non-additivity *N*	1	S_N	$MSN = S_N$	$F = \frac{MSN}{MSE}$
Error	$(I-1)(J-1)-1$	SSE (By subtraction)	MSE	
Total	$IJ-1$	TSS		