Analysis of Variance and Design of Experiments

Experimental Designs and Their Analysis

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Lecture 19

Randomized Block Design

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Randomized Block Design:

If a large number of treatments are to be compared, then a large number of experimental units are required.

This will increase the variation among the responses and CRD may not be appropriate to use.

In such a case when the experimental material is <u>not homogeneous</u> and there are *v* treatments to be compared, then it may be possible to group the experimental material into blocks of sizes *v* units.

Randomized Block Design:

- **Blocks are constructed such that the experimental units within a block are relatively homogeneous and resemble to each other more closely than the units in the different blocks.**
- **If there are *b* such blocks, we say that the blocks are at *b* levels.**
- **Similarly, if there are *v* treatments, we say that the treatments are at *v* levels.**
- **The responses from the *b* levels of blocks and *v* levels of treatments can be arranged in a two-way layout. The observed data set is arranged as follows:**

Randomized Block Design:

	Treatments (Factor B)							Block totals
		1	2		j		v	
Blocks (Factor A)	1	y_{11}	y_{12}	…	y_{1j}	…	y_{1v}	B_1
	2	y_{21}	y_{22}	…	y_{2j}	…	y_{2v}	B_2
	⋮	⋮	⋮		⋮		⋮	⋮
	i	y_{i1}	y_{i2}	…	y_{ij}	…	y_{iv}	B_i
	⋮		⋮		⋮		⋮	⋮
	b	y_{b1}	y_{b2}	…	y_{bj}	…	y_{bv}	B_b
Treatment totals		T_1	T_2	…	T_j	…	T_v	Grand total G

Randomized Block Design: Layout

A two-way layout is called a randomized block design (RBD) or a randomized complete block design (RCB) if, within each block, the *v* treatments are randomly assigned to *v* experimental units such that each of the *v*! ways of assigning the treatments to the units has the same probability of being adopted in the experiment and the assignment in different blocks are statistically independent.

The RBD utilizes the principles of design - randomization, replication and local control - in the following way:

Randomized Block Design: Randomization

1. **Randomization:**
   - **Number the *v* treatments 1,2,…,*v*.**
   - **Randomly allocate the *v* treatments to *v* experimental units in each block.**

Randomized Block Design:

		Treatments (Factor *B*)						Block totals
		1	2		j		v	
Blocks (Factor *A*)	1	y_{11}	y_{12}	…	y_{1j}	…	y_{1v}	B_1
	2	y_{21}	y_{22}	…	y_{2j}	…	y_{2v}	B_2
	⋮	⋮	⋮		⋮		⋮	⋮
	i	y_{i1}	y_{i2}	…	y_{ij}	…	y_{iv}	B_i
	⋮		⋮		⋮		⋮	⋮
	b	y_{b1}	y_{b2}	…	y_{bj}	…	y_{bv}	B_b
Treatment totals		T_1	T_2	…	T_j	…	T_v	Grand total G

Randomized Block Design: Replication

2. Replication

Since each treatment is appearing in each block, so every treatment will appear in all the blocks. So each treatment can be considered as if replicated the number of times as the number of blocks.

Thus in RBD, the number of blocks and the number of replications are same.

Randomized Block Design: Local Control

3. Local control

Local control is adopted in RBD in the following way:

- **First form the homogeneous blocks of the experimental units.**
- **Then allocate each treatment randomly in each block.**

The error variance now will be smaller because of homogeneous blocks and some variance will be parted away from the error variance due to the difference among the blocks.

Randomized Block Design: Example

Suppose there are 7 treatment denoted as $T_1, T_2, .., T_7$ **corresponding to 7 levels of a factor to be included in 4 blocks. So one possible layout of the assignment of 7 treatments to 4 different blocks in a RBD is as follows:**

Block 1	T_2	T_7	T_3	T_5	T_1	T_4	T_6
Block 2	T_1	T_6	T_7	T_4	T_5	T_3	T_2
Block 3	T_7	T_5	T_1	T_6	T_4	T_2	T_3
Block 4	T_4	T_1	T_5	T_6	T_2	T_7	T_3

Randomized Block Design: Analysis

Let

y_{ij} : Individual measurements of j^{th} treatment in i^{th} block,

$i = 1, 2, \ldots, b$, $j = 1, 2, \ldots, v$.

y_{ij}'s are independently distributed following $N(\mu + \beta_i + \tau_j, \sigma^2)$

where μ : **overall mean effect**

β_i : **i^{th} block effect**

τ_j : **j^{th} treatment effect**

such that

$$\sum_{i=1}^{b} \beta_i = 0, \ \sum_{j=1}^{v} \tau_j = 0 \ .$$

Randomized Block Design: Hypothesis

There are two null hypotheses to be tested:

- **related to the block effects**

$$H_{0B} : \beta_1 = \beta_2 = ... = \beta_b = 0.$$

- **related to the treatment effects**

$$H_{0T} : \tau_1 = \tau_2 = ... = \tau_v = 0.$$

Randomized Block Design:

The linear model in this case is a two-way model as

$$y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij},\ i = 1, 2, .., b;\ \ j = 1, 2, .., v$$

where ε_{ij} are identically and independently distributed random errors following a normal distribution with mean 0 and variance σ^2.

The tests of hypothesis can be derived using the likelihood ratio test or the principle of least squares. The use of likelihood ratio test has already been demonstrated earlier, so we now use the principle of least squares.

Randomized Block Design:

Minimizing

$$S = \sum_{i=1}^{b}\sum_{j=1}^{v}\varepsilon_{ij}^2 = \sum_{i=1}^{b}\sum_{j=1}^{v}(y_{ij} - \mu - \beta_i - \tau_j)^2$$

and solving the normal equation

$$\frac{\partial S}{\partial \mu} = 0, \; \frac{\partial S}{\partial \beta_i} = 0, \; \frac{\partial S}{\partial \tau_j} = 0 \text{ for all } i = 1, 2, .., b, \; j = 1, 2, .., v,$$

the least squares estimators are obtained as

$$\hat{\mu} = \bar{y}_{oo},$$

$$\hat{\beta}_i = \bar{y}_{io} - \bar{y}_{oo},$$

$$\hat{\tau}_j = \bar{y}_{oj} - \bar{y}_{oo}.$$

Randomized Block Design:

Using the fitted model, we can write

$$y_{ij} = \bar{y}_{oo} + (\bar{y}_{io} - \bar{y}_{oo}) + (\bar{y}_{oj} - \bar{y}_{oo}) + (y_{ij} - \bar{y}_{io} - \bar{y}_{oj} + \bar{y}_{oo}).$$

Squaring both sides and summing over *i* and *j* gives

$$\sum_{i=1}^{b}\sum_{j=1}^{v}(\bar{y}_{ij} - \bar{y}_{oo})^2 = v\sum_{i=1}^{b}(\bar{y}_{io} - \bar{y}_{oo})^2 + b\sum_{j=1}^{v}(\bar{y}_{oj} - \bar{y}_{oo})^2 + \sum_{i=1}^{b}\sum_{j=1}^{v}(y_{ij} - \bar{y}_{io} - \bar{y}_{oj} + \bar{y}_{oo})^2$$

or $\quad TSS \quad = \quad SSBl \quad + \quad SSTr \quad + \quad SSE$

with degrees of freedom partitioned as

$$bv - 1 = (b-1) + (v-1) + (b-1)(v-1).$$

The reason for the number of degrees of freedom for different sums of squares is the same as in the case of CRD.

Randomized Block Design:

Total SS: $TSS = \sum_{i=1}^{b}\sum_{j=1}^{v}(y_{ij} - \bar{y}_{oo})^2 = \sum_{i=1}^{b}\sum_{j=1}^{v} y_{ij}^2 - \frac{G^2}{bv}$

$G = \sum_{i=1}^{v}\sum_{j=1}^{n_i} y_{ij}$ **: Grand total,** $\frac{G^2}{bv}$**: Correction factor**

SS due to blocks: $SSBl = v\sum_{i=1}^{n}(\bar{y}_{io} - \bar{y}_{oo})^2 = \sum_{i=1}^{b}\frac{B_i^2}{b} - \frac{G^2}{bv}$

$B_i = \sum_{j=1}^{v} y_{ij}$: i^{th} **block total**

SS due to treatments: $SSTr = b\sum_{j=1}^{v}(\bar{y}_{oj} - \bar{y}_{oo})^2 = \sum_{j=1}^{v}\frac{T_j^2}{v} - \frac{G^2}{bv}$

$T_j = \sum_{i=1}^{b} y_{ij}$: j^{th} **treatment total**

SS due to error: $SSE = \sum_{i=1}^{b}\sum_{j=1}^{v}(y_{ij} - \bar{y}_{io} - \bar{y}_{oj} + \bar{y}_{oo})^2.$

Randomized Block Design:

Moreover,

$$(b-1)\frac{SSBl}{\sigma^2} \sim \chi^2(b-1)$$

$$(v-1)\frac{SSTr}{\sigma^2} \sim \chi^2(v-1)$$

$$(b-1)(v-1)\frac{SSE}{\sigma^2} \sim \chi^2(b-1)(v-1).$$

Under $H_{0B} : \beta_1 = \beta_2 = \ldots = \beta_b = 0,$

$$E(MSBl) = E(MSE)$$

and $SSBl$ **and** SSE **are independent, so**

$$F_{bl} = \frac{MSBl}{MSE} \sim F((b-1,(b-1)(v-1)).$$

Randomized Block Design:

Similarly, under $H_{0T}: \tau_1 = \tau_2 = \ldots = \tau_v = 0,$

$$E(MSTr) = E(MSE).$$

Also, $SSTr$ **and** SSE **are independent, so**

$$F_{Tr} = \frac{MSTr}{MSE} \sim F(v-1),(b-1)(v-1)).$$

Reject H_{0B} *if* $F_{be} > F_{\alpha}((b-1),(b-1)(v-1)$

Reject H_{0T} *if* $F_{Tr} > F_{\alpha}((v-1),(b-1)(v-1))$

Randomized Block Design:

If H_{0B} is accepted, then it indicates that the blocking is not necessary for future experimentation.

If H_{0T} is rejected then it indicates that the treatments are different.

Then the multiple comparison tests are used to divide the entire set of treatments into different subgroup such that the treatments in the same subgroup have the same treatment effect and those in the different subgroups have different treatment effects.

Randomized Block Design: ANOVA table

The analysis of variance table for RBD is

Source of variation	Degrees of freedom	Sum of squares	Mean squares	*F* - value
Blocks	$b-1$	$SSBl$	$MSBl$	$F_{bl} = \frac{MSBl}{MSE}$
Treatments	$v-1$	$SSTr$	$MSTr$	$F_{tr} = \frac{MSTr}{MSE}$
Errors	$(b-1)(v-1)$	SSE	MSE	
Total	$bv-1$	TSS		

Randomized Block Design:

$$E(MSBl) = E\left(\frac{SSBl}{b-1}\right) = \sigma^2 + \frac{v}{b-1}\sum_{i=1}^{b}\beta_i^2$$

$$E(MSTr) = E\left(\frac{SSTr}{v-1}\right) = \sigma^2 + \frac{b}{v-1}\sum_{j=1}^{v}\tau_j^2$$

$$E(MSE) = E\left(\frac{SSE}{(b-1)(v-1)}\right) = \sigma^2.$$