Analysis of Variance and Design of Experiments

Experimental Designs and Their Analysis

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Lecture 20

Basics in Latin Square Design

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Latin Square Design:

The treatments in the RBD are randomly assigned to *b* blocks such that each treatment must occur in each block rather than assigning them at random over the entire set of experimental units as in the CRD.

There are only two factors – block and treatment effects – which are taken into account and the total number of experimental units needed for complete replication are *bv* where *b* and *v* are the numbers of blocks and treatments respectively.

Latin Square Design:

If there are three factors and suppose there are *b*, *v* and *k* levels of each factor, then the total number of experimental units needed for a complete replication are *bvk*.

This increases the cost of experimentation and the required number of experimental units over RBD.

In Latin square design (LSD), the experimental material is divided into rows and columns, each having the same number of experimental units which is equal to the number of treatments.

Latin Square Design:

The treatments are allocated to the rows and the columns such that each treatment occurs once and only once in each row and in each column.

In order to allocate the treatment to the experimental units in rows and columns, we take help from Latin squares.

Latin Square:

A Latin square of order *p* is an arrangement of p^2 symbols in cells arranged in *p* rows and *p* columns such that each symbol occurs once and only once in each row and in each column.

For example, to write a Latin square of order 4,

- **choose four symbols – A, B, C and D.**
- **These letters are Latin letters which are used as symbols.**
- **Write them in a way such that each of the letters out of A, B, C and D occurs once and only once in each row and each column.**

Latin Square:

For example, as

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

This is a Latin square.

We consider first the following example to illustrate how a Latin square is used to allocate the treatments and in getting the response.

Latin Square: Example

Suppose different brands of petrol are to be compared with respect to the mileage per liter achieved in motor cars.

Important factors responsible for the variation in mileage are

- the difference between individual cars.

- the difference in the driving habits of drivers.

We have three factors – cars, drivers and petrol brands. Suppose we have

- 4 types of cars denoted as 1, 2, 3, 4.

- 4 drivers that are represented by a, b, c, d.

- 4 brands of petrol are indicated as A, B, C, D.

Latin Square:

Now the complete replication will require $4 \times 4 \times 4 = 64$ number of experiments.

We choose only 16 experiments.

To choose such 16 experiments, we take the help of the Latin square.

Suppose we choose the following Latin square:

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

Latin Square:

Write them in rows and columns and choose rows for drivers, columns for cars and letter for petrol brands.

Thus 16 observations are recorded as per this plan of treatment combination and further analysis is carried out.

Since such design is based on Latin square, so it is called as a Latin square design.

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

Latin Square:

		Cars			
		1	2	3	4
Drivers	a	A	B	C	D
	b	B	C	D	A
	c	C	D	A	B
	d	D	A	B	C

Driver "d" will use petrol C in car 4.

Driver "a" will use petrol A in car 1.

Driver "b" will use petrol C in car 2.

Latin Square:

Another choice of a Latin square of order 4 is :

C	B	A	D
B	C	D	A
A	D	C	B
D	A	B	C

This will again give a design different from the previous one.

The 16 observations will be recorded again but based on different treatment combinations.

Since we use only 16 out of 64 possible observations, so it is an incomplete 3-way layout in which each of the 3 factors – cars, drivers and petrol brands are at 4 levels and the observations are recorded only on 16 of the 64 possible treatment combinations.

Latin Square:

Thus in an LSD,

- **the treatments are grouped into replication in two-ways**
  - ❖ **once in rows and**
  - ❖ **and in columns,**
- **rows and columns variations are eliminated from the within treatment variation.**

Latin Square:

- **In RBD, the experimental units are divided into homogeneous blocks according to the blocking factor. Hence it eliminates the difference among blocks from the experimental error.**

- **In LSD, the experimental units are grouped according to two factors. Hence two effects (like as two block effects) are removed from the experimental error.**

- **So the error variance can be considerably reduced in LSD.**

Latin Square Design:

The LSD is an incomplete three-way layout in which each of the three factors, viz, rows, columns and treatments, is at v levels each and observations only on v^2 of the v^3 possible treatment combinations are taken.

Each treatment combination contains one level of each factor.

Latin Square Design:

The analysis of data in an LSD is conditional in the sense it depends on which Latin square is used for allocating the treatments. If the Latin square changes, the conclusions may also change.

We note that Latin squares play an important role is an LSD, so first we study more about these Latin squares before describing the analysis of variance.

Standard form of Latin square:

A Latin square is in the standard form if the symbols in the first row and first columns are in the natural order (Natural order means the order of alphabets like A, B, C, D,…).

Given a Latin square, it is possible to rearrange the columns so that the first row and first column remain in a natural order.

Standard form of Latin square: Example

Four standard forms of 4 x 4 Latin square are as follows.

A B C D B A D C C D B A D C A B	A B C D B C D A C D A B D A B C	A B C D B D A C C A D B D C B A	A B C D B A D C C D A B D C B A

For each standard Latin square of order *p*, the *p* rows can be permuted in *p*! ways.

Keeping a row fixed, vary and permute (*p* - 1) columns in (*p* - 1)! ways. So there are *p*!(*p* - 1)! different Latin squares.

Standard form of Latin square:

For illustration

Size of square	Number of standard squares	Value of $p!(p-1)!$	Total number of different squares
3 X 3	1	12	12
4 X 4	4	144	576
5 X 5	56	2880	161280
6 X 6	9408	86400	812851250

Conjugate and self conjugate:

Two standard Latin squares are called conjugate if the rows of one are the columns of other .

For example

A	B	C	D		A	B	C	D
B	C	D	A	and	B	C	D	A
C	D	A	B		C	D	A	B
D	A	B	C		D	A	B	C

are conjugate.

In fact, they are self conjugate.

A Latin square is called self conjugate if its arrangement in rows and columns are the same.

Transformation set:

A set of all the Latin squares obtained from a single Latin square by permuting its rows, columns and symbols is called a transformation set.

From a Latin square of order p, $p!(p-1)!$ different Latin squares can be obtained by making $p!$ permutations of columns and $(p - 1)!$ permutations of rows which leaves the first row in place.

Thus

Number of different Latin squares of order p in a transformation set	**=**	**$p!(p-1)!$ x number of standard Latin squares in the set**

Orthogonal Latin squares:

If two Latin squares of the same order but with different symbols are such that when they are superimposed on each other, every ordered pair of symbols (different) occurs exactly once in the Latin square, then they are called orthogonal.

Graeco-Latin square:

A pair of orthogonal Latin squares, one with Latin symbols and the other with Greek symbols form a Graeco-Latin square.

For example

$$\begin{array}{cccc} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{array} \qquad \begin{array}{cccc} \alpha & \beta & \gamma & \delta \\ \delta & \gamma & \beta & \alpha \\ \beta & \alpha & \delta & \gamma \\ \gamma & \delta & \alpha & \beta \end{array}$$

is a Graeco-Latin square of order 4.

Graeco Latin squares design enables to consider one more factor than the factors in Latin square design.

Graeco-Latin square: Example

In the earlier example, if there are 4 drivers, 4 cars, 4 types of petrol and each petrol has four varieties, as α, β, γ and δ, then Graeco-Latin square helps in deciding the treatment combination as follows:

	Cars				
		1	**2**	**3**	**4**
Drivers	**a**	$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
	b	$B\delta$	$A\gamma$	$D\beta$	$C\alpha$
	c	$C\beta$	$D\alpha$	$A\delta$	$B\gamma$
	d	$D\gamma$	$C\delta$	$B\alpha$	$A\beta$

Now

$A\alpha$ **means: Driver 'a' will use the** α **variant of petrol A in Car 1.**

$B\gamma$ **means: Driver 'c' will use the** β **variant of petrol B in Car 4**

and so on.

Mutually orthogonal Latin square:

A set of Latin squares of the same order is called a set of mutually orthogonal Latin square (or a hyper Graeco-Latin square) if every pair in the set is orthogonal.

The total number of mutually orthogonal Latin squares of order *p* is at most (*p* - 1).