Analysis of Variance and Design of Experiments

Incomplete Block Designs and Their Analysis

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Lecture 24

Estimation of Parameters in IBD

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Normal equations:

Minimizing $S = \sum_i \sum_j \sum_m \varepsilon_{ijm}^2$ **with respect to** β_i **and** τ_j**, we obtain the least-squares estimators of the parameters as follows:**

$$\frac{\partial S}{\partial \beta_i} = 0 \Rightarrow \sum_j \sum_m (y_{ijm} - \beta_i - \tau_j) = 0$$

$$B_i = \beta_i k_i + \sum_j \tau_j n_{ij} \qquad [b \textbf{ equations}] \qquad \textbf{(1)}$$

$$\frac{\partial S}{\partial \tau_j} = 0 \Rightarrow \sum_i \sum_m (y_{ijm} - \beta_i - \tau_j) = 0$$

$$V_j = \sum_i \beta_i n_{ij} + r_j \tau_j \qquad [v \textbf{ equations}] \qquad \textbf{(2)}$$

$$\sum_i \textbf{equation } (1) = \sum_j \textbf{equation } (2)$$

$$\sum_i B_i = \sum_j V_j$$

$$\sum_i \left(\sum_j \sum_m y_{ijm} \right) = \sum_j \left(\sum_i \sum_m y_{ijm} \right).$$

Estimation of parameters:

To obtain the estimates of the parameters, there are two options-

1. Using equation (1), eliminate β_i from equation (2) to estimate τ_j or

2. Using equation (2), eliminate τ_j from equation (1) to estimate β_i .

We consider first the approach 1., i.e., using equation (1), eliminate β_i from equation (2).

From equation (1),

$$\beta_i = \frac{1}{k_i}\left[B_i - \sum_{j=1}^{v} n_{ij}\tau_j\right].$$

Use it in (2) as follows.

Estimation of parameters and treatment totals:

We consider first the approach 1., i.e., using equation (1), eliminate β_i from equation (2).

From equation (1),

$$\beta_i = \frac{1}{k_i}\left[B_i - \sum_{j=1}^{v} n_{ij}\tau_j \right].$$

Use it in (2) as follows.

$$V_j = n_{1j}\beta_1 + \ldots + n_{bj}\beta_b + r_j\tau_j$$

$$= n_{1j}\left[\frac{1}{k_1}(B_1 - n_{11}\tau_1 - \ldots - n_{1v}\tau_{1v}) \right] + n_{2j}\left[\frac{1}{k_2}(B_2 - n_{21}\tau_1 - \ldots - n_{2v}\tau_v) \right] + \ldots$$

$$+ n_{bj}\left[\frac{1}{k_b}(B_b - n_{b1}\tau_1 - \ldots - n_{bv}\tau_v) \right] + r_j\tau_j$$

Adjusted treatment totals:

$$V_j = n_{1j}\left[\frac{1}{k_1}(B_1 - n_{11}\tau_1 - \ldots - n_{1v}\tau_{1v})\right] + n_{2j}\left[\frac{1}{k_2}(B_2 - n_{21}\tau_1 - \ldots - n_{2v}\tau_v)\right] + \ldots$$

$$+ n_{bj}\left[\frac{1}{k_b}(B_b - n_{b1}\tau_1 - \ldots - n_{bv}\tau_v)\right] + r_j\tau_j$$

or $$V_j - \sum_{i=1}^{b}\frac{n_{ij}B_i}{k_i} = \tau_1\left[-\frac{n_{11}n_{1j}}{k_1}\ldots - \frac{n_{b1}n_{bj}}{k_b}\right] + \ldots + \tau_v\left[-\frac{n_{1v}n_{1j}}{k_1}\ldots - \frac{n_{bv}n_{bj}}{k_b}\right] + r_j\tau_j$$

or

$$Q_j = \tau_1\left[-\frac{n_{11}n_{1j}}{k_1}\ldots - \frac{n_{b1}n_{bj}}{k_b}\right] + \ldots + \tau_v\left[-\frac{n_{1v}n_{1j}}{k_1}\ldots - \frac{n_{bv}n_{bj}}{k_b}\right] + r_j\tau_j, \quad j = 1\ldots v$$

are Q_j called adjusted treatment totals.

Normal equations:

Write

$$Q_j = \tau_1\left[-\frac{n_{11}n_{1j}}{k_1}\dots-\frac{n_{b1}n_{bj}}{k_b}\right]+\dots+\tau_v\left[-\frac{n_{1v}n_{1j}}{k_1}\dots-\frac{n_{bv}n_{bj}}{k_b}\right]+r_j\tau_j,\quad j=1,2,\dots,v.$$

as

$$Q_j = C_{j1}\tau_1 + C_{j2}\tau_2 + \dots + C_{jv}\tau_v$$

where

$$C_{jj} = r_j - \frac{n_{1j}^2}{k_1} - \frac{n_{2j}^2}{k_2} - \dots - \frac{n_{bj}^2}{k_b}$$

$$C_{jj'} = -\frac{n_{1j}n_{1j'}}{k_1} - \frac{n_{2j}n_{2j'}}{k_2} - \dots - \frac{n_{bj}n_{bj'}}{k_b};\quad j \neq j',\ j=1,2,..,v.$$

Normal equations:

The $v \times v$ matrix $C = ((C_{jj'})),\ j = 1,2,...,v;\ j' = 1,2,...,v$ with C_{jj} as diagonal elements and $C_{jj'}$ as off-diagonal elements is called the *C*-matrix of the incomplete block design.

***C* matrix is symmetric.**

Its row sum and columns sums are zero. (proved later).

Normal equations:

Rewrite

$$Q_j = \tau_1\left[-\frac{n_{11}n_{1j}}{k_1}\ldots-\frac{n_{b1}n_{bj}}{k_b}\right]+\ldots+\tau_v\left[-\frac{n_{1v}n_{1j}}{k_1}\ldots-\frac{n_{bv}n_{bj}}{k_b}\right]+r_j\tau_j,\quad j=1,2,\ldots,v.$$

as

$$Q = C\tau.$$

This equation is called as reduced normal equations where

$$Q' = (Q_1, Q_2, .., Q_v),\ \ \tau' = (\tau_1, \tau_2, \ldots, \tau_v)$$

Equations (1) and (2) are EQUIVALENT.

Alternative presentation in matrix notations:

Now let us try to represent and translate the same algebra in matrix notations.

Let $E_{mn} : m \times n$ **matrix whose all elements are unity.**

$N = (n_{ij})$ **is** $b \times v$ **matrix called as incidence matrix.**

$$k_i = \sum_{j=1}^{v} n_{ij}$$

$$r_j = \sum_{i=1}^{b} n_{ij}$$

$$n = \sum_i \sum_j n_{ij}$$

$$E_{1b} N = (r_1, r_2, ..., r_v) = r'$$

$$N E_{v1} = (k_1, k_2, ..., k_b)' = k.$$

Alternative presentation in matrix notations:

For illustration, we verify one of the relationships as follows.

$$E_{1b} = (1,1,\ldots,1)_{1\times b}$$

$$N = \begin{pmatrix} n_{11} & n_{12} \cdots n_{1v} \\ n_{21} & n_{22} \cdots n_{2v} \\ \vdots & \vdots \quad \ddots \vdots \\ n_{b1} & n_{b2} \cdots n_{bv} \end{pmatrix}_{b\times v}$$

$$E_{1b}N = (1,1,\ldots,1)_{1\times b} \begin{pmatrix} n_{11} & n_{12} \cdots n_{1v} \\ n_{21} & n_{22} \cdots n_{2v} \\ \vdots & \vdots \quad \ddots \vdots \\ n_{b1} & n_{b2} \cdots n_{bv} \end{pmatrix}_{b\times v} = \left(\sum_{i=1}^{b} n_{i1}, \sum_{i=1}^{b} n_{i2}, \ldots, \sum_{i=1}^{b} n_{iv} \right)$$

$$= (r_1, r_2, \ldots, r_v) = r'.$$

Estimation of parameters:

It is now clear that the treatment and blocks are not estimable as such as in the case of complete block designs. Note that we have not made any assumption like $\sum_i \alpha_i = \sum_j \beta_j = 0$ also.

Now we introduce the general mean effect (denoted by μ) in the linear model and carry out further analysis on the same lines as earlier.

Estimation of parameters:

Consider the model

$$y_{ijm} = \mu + \beta_i + \tau_j + \varepsilon_{ijm}, \quad i = 1, 2, ..., b; \; j = 1, 2, ..., v; m = 0, 1, .., n_{ij}.$$

The normal equations are obtained by minimizing $S = \sum_i \sum_j \sum_m \varepsilon_{ijm}^2$ **with respect to the parameters** μ, β_i **and** τ_j **and solving them, we can obtain the least-squares estimators of the parameters.**

Minimizing $S = \sum_i \sum_j \sum_m \varepsilon_{ijm}^2$ **with respect to the parameters** μ, β_i **and** τ_j**, the normal equations are obtained as**

$$n\hat{\mu} + \sum_i n_{io}\hat{\beta}_i + \sum_j n_{oj}\hat{\tau}_j = G$$

$$n_i\hat{\mu} + n_{io}\hat{\beta}_i + \sum_j n_{ij}\hat{\tau}_j = B_i \qquad i = 1, ..., b$$

$$n_{oj}\hat{\mu} + n_{oj}\hat{\tau}_j + \sum_i n_{ij}\hat{\beta}_i = V_j \qquad j = 1, ..., v.$$

Estimation of parameters:

Now we write these normal equations in matrix notations.

Denote $\beta = Col(\beta_1, \beta_2, ..., \beta_b)$

$$\tau = Col(\tau_1, \tau_2, ..., \tau_v)$$

$$B = Col(B_1, B_2, ..., B_b)$$

$$V = Col(V_1, V_2 ..., V_v)$$

$N = ((n_{ij}))$: **incidence matrix of order** $b \times v$

where *Col* (.) denotes the column vector. Let

$K = diag(k_1, .., k_b) : b \times b$ **diagonal matrix**

$R = diag(r_1, .., r_v) : v \times v$ **diagonal matrix.**

Estimation of parameters:

$$n\hat{\mu} + \sum_i n_{io}\hat{\beta}_i + \sum_j n_{oj}\hat{\tau}_j = G$$

$$n_i\hat{\mu} + n_{io}\hat{\beta}_i + \sum_j n_{ij}\hat{\tau}_j = B_i \qquad i = 1, \ldots, b$$

$$n_{oj}\hat{\mu} + n_{oj}\hat{\tau}_j + \sum_i n_{ij}\hat{\beta}_i = V_j \qquad j = 1, \ldots, v.$$

These (*b* + *v* +1) normal equations can be written as

$$\begin{pmatrix} G \\ B \\ V \end{pmatrix} = \begin{pmatrix} n & E_{1b}K & E_{1v}R \\ KE_{b1} & K & N \\ RE_{v1} & N' & R \end{pmatrix} \begin{pmatrix} \hat{\mu} \\ \hat{\beta} \\ \hat{\tau} \end{pmatrix}. \qquad (*)$$

Since we are presently interested in the testing of hypothesis related to the treatment effects, so we eliminate the block effects $\hat{\beta}$ **to estimate the treatment effects.**

Estimation of parameters:

Rewrite equation (*) as

$$G = n\hat{\mu} + E_{1b}K\hat{\beta} + E_{1v}R\hat{\tau} \qquad \text{(i)}$$

$$B = KE_{b1}\hat{\mu} + K\hat{\beta} + N\hat{\tau} \qquad \text{(ii)}$$

$$V = RE_{v1}\hat{\mu} + N'\hat{\beta} + R\hat{\tau} \qquad \text{(iii)}$$

These are called as 'reduced normal equations' or 'reduced intrablock equations'.

Premultiply equation (ii) by $N'K^{-1}$ **as**

$$N'K^{-1}B = N'K^{-1}KE_{b1}\hat{\mu} + N'K^{-1}K\hat{\beta} + N'K^{-1}N\hat{\tau}$$

and subtract it from equation (iii) as

$$V - N'K^{-1}B = (RE_{v1} - N'K^{-1}KE_{b1})\hat{\mu} + (N' - N'K^{-1}K)\hat{\beta} + (R - N'K^{-1}N)\hat{\tau}$$

or

$$V - N'K^{-1}B = [R - N'K^{-1}N]\hat{\tau}.$$

Estimation of parameters:

$$V - N'K^{-1}B = [R - N'K^{-1}N]\hat{\tau}.$$

The reduced normal equation in the treatment effects can be written as

$$Q = C\hat{\tau}$$

where

$$Q = V - N'K^{-1}B$$

$$C = R - N'K^{-1}N.$$

The vector *Q* is called as the vector of adjusted treatment totals since it contains the treatment totals adjusted for the block effects, the matrix *C* is called as *C*-matrix.

Estimation of parameters:

The C matrix is symmetric and its row sums and columns sums are zero.

To show that row sum is zero in C matrix, we proceed as follows:

Row sum:
$$\begin{aligned} CE_{v1} &= RE_{v1} - N'K^{-1}NE_{v1} \\ &= (r_1, r_2, ..., r_v)' - N'K^{-1}k = (r_1, r_2, ..., r_v)' - N'E_{b1} \\ &= r - r = 0 \end{aligned}$$

Similarly, the column sum can also be shown to be zero.