Analysis of Variance and Design of Experiments

Incomplete Block Designs and Their Analysis

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Lecture 26

Properties of Treatment and Block Totals

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Model:

Consider the model

$$y_{ijm} = \mu + \beta_i + \tau_j + \varepsilon_{ijm}, \quad i = 1, 2, ..., b; \ j = 1, 2, ..., v; m = 0, 1, .., n_{ij}.$$

The normal equations are obtained by minimizing $S = \sum_i \sum_j \sum_m \varepsilon_{ijm}^2$

with respect to the parameters μ, β_i **and** τ_j **and solving them,**

we can obtain the least-squares estimators of the parameters.

These (*b* + *v* +1) normal equations can be written as

$$\begin{pmatrix} G \\ B \\ V \end{pmatrix} = \begin{pmatrix} n & E_{1b}K & E_{1v}R \\ KE_{b1} & K & N \\ RE_{v1} & N' & R \end{pmatrix} \begin{pmatrix} \hat{\mu} \\ \hat{\beta} \\ \hat{\tau} \end{pmatrix}. \qquad (*)$$

Reduced Normal Equations:

The reduced normal equation in the treatment effects after adjusting for the block effects are

$$Q = C\hat{\tau}$$

where $Q = V - N'K^{-1}B,\qquad C = R - N'K^{-1}N.$

The reduced normal equation in the block effects after adjusting for the treatment effects are

$$P = D\hat{\beta}$$

where $P = B - NR^{-1}V,\qquad D = K - NR^{-1}N'.$

Orthogonality of adjusted treatment total and unadjusted block totals. :

Theorem: **The adjusted treatment totals are orthogonal to the block totals.**

Proof: **It is enough to prove that** $Cov(B_i, Q_j) = 0$ for all i, j.

Now

$$Cov(B_i, Q_j) = Cov\left[B_i, V_j - \sum_i \left(\frac{n_{ij}}{k_i}\right) B_i\right]$$

$$= Cov(B_i, V_j) - \frac{n_{ij}}{k_i} Var(B_i)$$

because the block totals are mutually orthogonal, see how:

Orthogonality of adjusted treatment total and unadjusted block totals. :

For $y_{11}, y_{12}, \ldots, y_{1v}$, **the block total** $B_1 = \sum_{j=1}^{v} y_{1j}$.

For $y_{21}, y_{22}, \ldots, y_{2v}$, **the block total** $B_2 = \sum_{j=1}^{v} y_{2j}$.

$Var(B_1) = \sum_{j=1}^{v} Var(y_{1j}) = v\sigma^2$ **as** $Cov(y_{1j}, y_{1k}) = 0$ **for** $j \neq k$

$Var(B_2) = \sum_{j=1}^{v} Var(y_{2j}) = v\sigma^2$ **as** $Cov(y_{2j}, y_{2k}) = 0$ **for** $j \neq k$

$Var(B_1) + Var(B_2) = 2v\sigma^2$ **as** $Cov(y_{1j}, y_{2k}) = 0$ **for** $j \neq k$

$\Rightarrow$ B_1 **and** B_2 **are mutually orthogonal as all** y_{ij}**'s are independent.**

Orthogonality of adjusted treatment total and unadjusted block totals. :

As B_i and V_j have n_{ij} observations in common and the observations are mutually independent, so

$$Cov(B_i, V_j) = n_{ij}\sigma^2$$

$$Var(B_i) = k_i\sigma^2$$

Thus $Cov(B_i, Q_j) = n_{ij}\sigma^2 - \frac{n_{ij}}{k_i} k_i\sigma^2 = 0$

Hence proved.

$E(Q) = C\tau$, $Var(Q) = \sigma^2 C$

Theorem:

$$E(Q) = C\tau$$

$$Var(Q) = \sigma^2 C$$

Proof:

$$Q_j = V_j - \left[\frac{n_{1j}B_1}{k_1} + \ldots + \frac{n_{bj}B_b}{k_b}\right] = V_j - \sum_{i=1}^{b} \frac{n_{ij}B_i}{k_i}$$

$$E(Q_j) = E(V_j) - \sum_{i=1}^{b} \frac{n_{ij}}{k_i} E(B_i)$$

Find both the expectations as follows:

$$E(V_j) = \sum_i \sum_m E(y_{ijm}) = \sum_i \sum_m E(\mu + \beta_i + \tau_j + \varepsilon_{ijm})$$

$$= \mu \sum_i n_{ij} + \sum_i \beta_i n_{ij} + \tau_j \sum_i n_{ij}$$

$$= \mu r_j + \sum_i \beta_i n_{ij} + \tau_j r_j$$

$E(Q) = C\tau$, $Var(Q) = \sigma^2 C$

$$E(B_i) = \sum_j \sum_m E(y_{ijm})$$

$$= \sum_j \sum_m E(\mu + \beta_i + \tau_j + \varepsilon_{ijm})$$

$$= \sum_j \sum_m (\mu + \beta_i + \tau_j)$$

$$= \mu k_i + \beta_i k_i + \sum_j \tau_j n_{ij}$$

$$\sum_{i=1}^{b} \frac{n_{ij}}{k_i} E(B_i) = \sum_{i=1}^{b} \frac{n_{ij}}{k_i} \left[\mu k_i + \beta_i k_i + \sum_j \tau_j n_{ij} \right]$$

$$= \mu r_j + \sum_i \beta_i n_{ij} + \sum_i \frac{n_{ij}}{k_i} (\sum_j \tau_j n_{ij}).$$

Thus substituting these expressions in $E(Q_j)$, **we have**

$E(Q) = C\tau$, $Var(Q) = \sigma^2 C$

$$E(Q_j) = r_j\tau_j - \sum_i \frac{n_{ij}}{k_i}\left(\sum_j \tau_j n_{ij}\right)$$

$$= \left(r_i - \sum_i \frac{n_{ij}^2}{k_i}\right)\tau_j - \sum_i \frac{n_{ij}}{k_i} \sum_{j(\neq \ell)} n_{i\ell}\tau_\ell = c_{jj}\tau_j + \sum_{j'(\neq \ell)} c_{\ell j'}\tau_{j'}$$

Further, substituting $E(Q_j)$, **in** $E(Q) = (E(Q_1), E(Q_2), \ldots, E(Q_b))'$ **, we get**

$$E(Q) = C\tau$$

$E(Q) = C\tau$, $Var(Q) = \sigma^2 C$

Next

$$Var(Q) = \begin{pmatrix} Var(Q_1) & Cov(Q_1, Q_2) & \cdots & Cov(Q_1, Q_v) \\ Cov(Q_2, Q_1) & Var(Q_2) & \cdots & Cov(Q_2, Q_v) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(Q_v, Q_1) & Cov(Q_v, Q_2) & \cdots & Var(Q_v) \end{pmatrix}$$

$$Var(Q_j) = Var[V_j - \sum_i \frac{n_{ij}}{k_i} B_i]$$

$$= Var(V_j) + \sum_i \left(\frac{n_{ij}}{k_i}\right)^2 Var(B_i) - 2\sum_i \frac{n_{ij}}{k_i} Cov(V_j, B_i),$$

$E(Q) = C\tau$, $Var(Q) = \sigma^2 C$

Note that $Var(V_j) = Var(\sum_i \sum_m y_{ijm}) = r_j \sigma^2$

$$Var(B_i) = Var(\sum_j \sum_m y_{ijm}) = k_i \sigma^2$$

$$Cov(V_j, B_i) = Cov\left(\sum_i \sum_m y_{ijm}, \sum_j \sum_m y_{ijm} \right) = n_{ij} \sigma^2$$

$$Var(Q_j) = r_j \sigma^2 + \sum_i \left(\frac{n_{ij}}{k_i} \right)^2 k_i \sigma^2 - 2 \sum_i \left(\frac{n_{ij}}{k_i} \right) n_{ij} \sigma^2$$

$$= r_j \sigma^2 - \sum_i \frac{n_{ij}^2}{k_i} \sigma^2 - 2 \sum_i \left(\frac{n_{ij}^2}{k_i} \right) \sigma^2 = r_j \sigma^2 - \sum_i \left(\frac{n_{ij}^2}{k_i} \right) \sigma^2$$

$$= c_{jj} \sigma^2 .$$

$E(Q) = C\tau$, $Var(Q) = \sigma^2 C$

$$Cov(Q_j, Q_\ell) = Cov\left[V_j - \sum_i \frac{n_{ij}}{k_i} B_i,\ V_\ell - \sum_i \frac{n_{i\ell}}{k_i} B_i\right]$$

$$= Cov(V_j, V_\ell) - \sum_i \frac{n_{i\ell}}{k_i} Cov(V_j, B_i) - \sum_i \frac{n_{ij}}{k_i} Cov(B_i, V_\ell) + \sum_i \frac{n_{ij} n_{i\ell}}{k_i^2} Cov(B_i, B_i)$$

$$= 0 - \sum_i \frac{n_{i\ell}}{k_i} Cov(V_j, B_i) - \sum_i \frac{n_{ij}}{k_i} Cov(B_i, V_\ell) + \sum_i \frac{n_{ij} n_{i\ell}}{k_i^2} Var(B_i)$$

$$= -\sum_i \left(\frac{n_{i\ell} n_{ij}}{k_i}\right)\sigma^2 - \sum_i \left(\frac{n_{ij} n_{i\ell}}{k_i}\right)\sigma^2 + \sum_i \frac{n_{ij} n_{i\ell}}{k_i^2} k_i \sigma^2 = c_{j\ell}\sigma^2$$

Substituting the terms of

$Var(Q_j) = c_{jj}\sigma^2$ **and** $Cov(Q_j, Q_l) = c_{jl}\sigma^2$ **in** $Var(Q)$,

we get $Var(Q) = C\sigma^2$.

Hence proved.

[*Note: We will prove this result using the matrix approach later*].

Covariance matrix of adjusted treatment totals:

Consider

$$Z = \begin{pmatrix} V \\ B \end{pmatrix}$$ **with** $b+v$ **variable.**

We can express

$$Q = V - N'K^{-1}B$$

$$= [I \quad -N'K^{-1}] \begin{bmatrix} V \\ B \end{bmatrix}$$

$$= [I \quad -N'K^{-1}]Z.$$

So

$$Cov(Q) = [I \quad -N'K^{-1}] \, Cov(Z) \begin{bmatrix} I' \\ (-N'K^{-1})' \end{bmatrix}.$$

Now we find

$$Cov(Z) = \begin{pmatrix} Var(V) & Cov(VB) \\ Cov(BV) & Var(B) \end{pmatrix}$$

Covariance matrix of adjusted treatment totals:

Since B_i and V_j have n_{ij} observations in common and the observations are mutually independent, so

$$Cov(B_i, V_j) = n_{ij}\sigma^2$$

$$Var(B_i) = k_i\sigma^2$$

$$Var(V_j) = r_j\sigma^2.$$

Thus

$$Cov(Z) = \begin{pmatrix} R & N' \\ N & K \end{pmatrix}\sigma^2$$

$$Cov(Q) = [I \quad -N'K^{-1}]\begin{bmatrix} R & N' \\ N & K \end{bmatrix}\begin{bmatrix} I \\ -K^{-1}N \end{bmatrix}\sigma^2$$

$$= [R - N'K^{-1}N \quad N' - N']\begin{bmatrix} I \\ -K^{-1}N \end{bmatrix}\sigma^2$$

$$= (R - N'K^{-1}N)\sigma^2$$

$$= C\sigma^2.$$

Covariance matrix of adjusted treatment totals:

Next, we show that $Cov(B,Q)=0$

$$Cov(B,Q) = Cov(B,V) - Cov(B, N'K^{-1}B)$$

$$= Cov(B,V) - Var(B)K^{-1}N$$

$$= N\sigma^2 - KK^{-1}N\sigma^2$$

$$= 0.$$

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Now we illustrate another approach to find the expectations etc. in the set up of an incomplete block design.

We have now learnt three approaches- the classical approach based on summations, the approach based on matrix theory and this new approach which is also based on the matrix theory.

We can choose any of the approaches.

The objective here is to let the reader know these different approaches.

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Rewrite the linear model

$$y_{ijm} = \mu + \beta_i + \tau_j + \varepsilon_{ijm}, \quad i = 1, 2, ..., b; \; j = 1, 2, ..., v; m = 0, 1, .., n_{ij}.$$

as $y = \mu E_{n1} + D_1^{'}\tau + D_2^{'}\beta + \varepsilon$

where $\tau = (\tau_1, \tau_2, ..., \tau_v)'$, $\quad \beta = (\beta_1, \beta_2, ..., \beta_b)'$

Since *B_i* and *V_j* have *n_{ij}* observations in common and the observations are mutually independent, so denote

$D_1 : v \times n$ **matrix of treatment effect versus *N*, i.e., (*i*, *j*)th element of this matrix is given by**

$$D_1 = \begin{cases} 1 & \text{if } j^{th} \text{ observation comes from } i^{th} \text{ treatment} \\ 0 & \text{otherwise} \end{cases}$$

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Similarly, we can find

$D_2 : b \times n$ **matrix of block effect versus *N*, i.e., (*i*, *j*)th element of this matrix is given by**

$$D_2 = \begin{cases} 1 & \text{if } j^{th} \text{ observation comes from } i^{th} \text{ block} \\ 0 & \text{otherwise.} \end{cases}$$

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Following results can be verified:

$$D_1 D_1^{'} = R = diag(r_1, r_2, ..., r_v),$$

$$D_2 D_2^{'} = K = diag(k_1, k_2, ..., k_b),$$

$$D_2 D_1^{'} = N \quad \text{or} \quad D_1 D_2^{'} = N'$$

$$D_1 E_{n1} = (r_1, r_2, ..., r_v)'$$

$$D_2 E_{n1} = (k_1, k_2, ..., k_b)'$$

$$D_1^{'} E_{v1} = E_{n1} = D_2^{'} E_{b1}.$$

In earlier notations,

$$V = (V_1, ..., V_v)' = D_1 y$$

$$B = (B_1, ..., B_b)' = D_2 y$$

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Express *Q* in terms of D_1 and D_2 as

$$Q = V - N'K^{-1}B$$

$$= \left[D_1 - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 \right] y.$$

Then

$$E(Q) = \left[D_1 - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 \right] E(y)$$

$$= \left[D_1 - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 \right] (\mu E_{n1} + D_1^{'} \tau + D_2^{'} \beta)$$

$$= \left[D_1 E_{n1} - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 E_{n1} \right] \mu + \left[D_1 D_1^{'} - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 D_1^{'} \right] \tau$$

$$+ \left[D_1 D_2^{'} - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 D_2^{'} \right] \beta$$

$$= \left[(r_1, r_2, ..., r_v)' - N'K^{-1}(k_1, ..., k_b)' \right] \mu + \left[R - N'K^{-1}N \right] \tau + \left[N' - N'K^{-1}K \right] \beta.$$

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Since $N'K^{-1}(k_1,\ldots,k_b)' = N' diag\left(\frac{1}{k_1},\frac{1}{k_2},\ldots,\frac{1}{k_b}\right)\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_b \end{pmatrix}$

$$= N'\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n_{11} & n_{21} \ldots n_{b1} \\ n_{12} & n_{22} \ldots n_{b2} \\ \vdots & \vdots \ \ddots \ \vdots \\ n_{iv} & n_{2v} \ldots n_{bv} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \left(\sum_{i=1}^{b} n_{i1},\ \sum_{i=1}^{b} n_{i2},\ldots,\sum_{i=1}^{b} n_{iv}\right)' = (r_1, r_2, \ldots, r_v)'.$$

Thus

$$N' - N'K^{-1}K = \left(r_1, r_2, \ldots, r_v\right)' - N'K^{-1}(k_1,\ldots,k_b) = 0$$

and so

$$E(Q) = \left[R - N'K^{-1}N\right]\tau = C\tau.$$

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Next

$$Var(Q) = D_1\left[I - D_2'(D_2D_2')^{-1}D_2\right]Var(y)\left[I - D_2'(D_2D_2')^{-1}D_2\right]D_1'$$

$$= \sigma^2 D_1\left[I - D_2'(D_2D_2')^{-1}D_2\right]D_1'$$

$$= \sigma^2\left[D_1D_1' - D_1D_2'(D_2D_2')^{-1}D_2D_1'\right]$$

$$= \sigma^2\left[R - N'K^{-1}N\right]$$

$$= \sigma^2 C.$$

Note that $\left[I - D_2'(D_2D_2')^{-1}D_2\right]$ **is an idempotent matrix.**

Alternative approach to prove $E(P) = D\beta$, $Var(P) = \sigma^2 D$:

Similarly, we can prove for $P = B - NR^{-1}V, \quad D = K - NR^{-1}N'$

Theorem: $E(P) = D\beta, \quad Var(P) = \sigma^2 D$

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Theorem: $E(P) = D\beta, \quad Var(P) = \sigma^2 D$

Proof: $D = K - NR^{-1}N'$

$$P = B - NR^{-1}V = D_2[I - D_1R^{-1}D_1]y = D_2[I - D_1^{'}(D_1D_1^{'})^{-1}D_1]y$$

$$E(P) = D_2[I_1 - D_1^{'}(D_1D_1^{'})^{-1}D_1](\mu E_{n1} + D_1^{'}\tau + D_2^{'}\beta)$$

$$= [D_2E_{n1} - D_2D_1^{'}R^{-1}D_1E_{n1}]\mu + [D_2D_1^{'} - D_2D_1^{'}R^{-1}D_1D_1^{'}]\tau$$

$$+ [D_2D_1^{'} - D_2D_1^{'}R^{-1}D_1D_2^{'}]\beta$$

$$= [(k_1, k_2, ..., k_b)' - NR^{-1}(r_1, r_2, ..., r_v)']\mu + [N - NR^{-1}R]\tau + [K - NR^{-1}N']\beta$$

$$= [(k_1, k_2, ..., k_b)' - NE_{v1}]\mu + 0 + D\beta$$

$$= [(k_1, k_2, ..., k_b)' - (k_1, k_2, ..., k_b)']\mu + D\beta$$

$$= D\beta$$

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Next

$$
\begin{aligned}
Var(P) &= \sigma^2 D_2[I - D_1'(D_1D_1')^{-1}D_1]D_2' \\
&= \sigma^2[D_2D_2' - D_2D_1'(D_1D_1')^{-1}D_1D_2'] \\
&= \sigma^2[K - NR^{-1}N'] = \sigma^2 D.
\end{aligned}
$$

Note that $[I - D_1'(D_1D_1')^{-1}D_1]$ **is an idempotent matrix.**

Alternative approach to prove $E(Q) = C\tau$, $Var(Q) = \sigma^2 C$:

Alternatively, we can also find *Var*(*P*) as follows:

$$P = (I \quad -NR^{-1})\begin{pmatrix} B \\ V \end{pmatrix} = (I \quad -NR^{-1})Z \quad \text{where } Z = (B, \quad V)'$$

$$\begin{aligned}
Var(P) &= (I \quad -NR^{-1})Cov(Z)\begin{pmatrix} I \\ -R^{-1}N' \end{pmatrix} \\
&= (I \quad -NR^{-1})\begin{pmatrix} K & N \\ N' & R \end{pmatrix}\begin{pmatrix} I \\ -R^{-1}N' \end{pmatrix}\sigma^2 \\
&= (K - NR^{-1}N' \quad N - NR^{-1}R)\begin{pmatrix} I \\ -R^{-1}N' \end{pmatrix}\sigma^2 \\
&= \left(K - NR^{-1}N'\right)\sigma^2 \\
&= D\sigma^2
\end{aligned}$$