Analysis of Variance and Design of Experiments

Incomplete Block Designs and Their Analysis

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Lecture 27

More Properties of Treatment and Block Totals

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Properties of incomplete block designs:

Now we consider some properties of incomplete block designs.

Lemma: ***b*** **+** ***rank*****(*****C*****) =** ***v*** **+** ***rank*****(*****D*****)** **.**

Proof: Consider (*b* + *v*) x (*b* + *v*) matrix

$$A = \begin{bmatrix} K & N \\ N' & R \end{bmatrix}$$

$$\begin{pmatrix} G \\ B \\ V \end{pmatrix} = \begin{pmatrix} n & E_{1b}K & E_{1v}R \\ KE_{b1} & K & N \\ RE_{v1} & N' & R \end{pmatrix} \begin{pmatrix} \hat{\mu} \\ \hat{\beta} \\ \hat{\tau} \end{pmatrix}$$

Note that *A* is a submatrix of the coefficient matrix of normal equations

Using the result that the rank of a matrix does not change by the pre-multiplication of the nonsingular matrix, consider the following matrices:

$$M = \begin{bmatrix} I_b & 0 \\ -N'K^{-1} & I_v \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} I_b & 0 \\ -R^{-1}N' & I_v \end{bmatrix}.$$

Properties of incomplete block designs:

M **and** ***S*** **are nonsingular, so we have**

***rank*(*A*) = *rank*(*MA*) = *rank*(*AS*).**

Now

$$MA = \begin{bmatrix} I_b & 0 \\ -N'K^{-1} & I_v \end{bmatrix} \begin{bmatrix} K & N \\ N' & R \end{bmatrix} = \begin{bmatrix} K & N \\ 0 & C \end{bmatrix}$$

$$AS = \begin{bmatrix} D & N \\ 0 & R \end{bmatrix}$$

Thus

$$rank \begin{bmatrix} K & N \\ 0 & C \end{bmatrix} = rank \begin{bmatrix} D & N \\ 0 & R \end{bmatrix}$$

or

$$rank(K) + rank(C) = rank(D) + rank(R)$$

or

$b + rank(C) = v + rank(D).$ **Hence proved.**

Properties of incomplete block designs:

Remark: $C : v \times v$ **and** $D : b \times b$ **are symmetric matrices.**

One can verify that $CE_{v1} = 0$ **and** $DE_{b1} = 0$

Thus $rank(C) \leq v - 1$

$$rank(D) \leq b - 1.$$

Properties of incomplete block designs:

Lemma: **If $rank(C) = v - 1$, then all blocks and treatment contrasts are estimable.**

Proof: **If $rank(C) = v - 1$, it is obvious that all the treatment contrasts are estimable.**

Using the result from the lemma $b + rank(C) = v + rank(D)$,

we have $rank(D) + v = rank(C) + b$

$$= v - 1 + b$$

Thus $rank(D) = b - 1$.

Thus all the block contrasts are also estimable.

Orthogonality of Q and P:

Now we explore the conditions under which *Q* and *P* can be orthogonal.

$$Q = V - N'K^{-1}B \quad = (D_1 - D_1 D_2' K^{-1} D_2)y$$

$$P = B - NR^{-1}V \quad = (D_2 - D_2 D_1' R^{-1} D_1)y$$

$$Cov(Q,P) = (D_1 - D_1 D_2' K^{-1} D_2)(D_2 - D_2 D_1' R^{-1} D_1)'\sigma^2$$

$$= (D_1 D_2' - D_1 D_2' R^{-1} D_1 D_2' - D_1 D_2' K^{-1} D_2 D_2' + D_1 D_2' K^{-1} D_2 D_1' R^{-1} D_1 D_2')\sigma^2$$

$$= (N' - RR^{-1}N' - N'K^{-1}K + N'K^{-1}NR^{-1}N')\sigma^2$$

$$= (N'K^{-1}NR^{-1}N' - N')\sigma^2$$

Orthogonality of Q and P:

Q and P (or equivalently Q_i and P_j) are orthogonal when

$$Cov(Q,P)=0$$

or $N'K^{-1}N\ R^{-1}N'-N'=0$ **(i)**

$\Rightarrow (R-C)R^{-1}N'-N'=0$ **(Using** $C=R-N'K^{-1}N$**)**

$\Rightarrow CR^{-1}N'=0$ **(ii)**

or equivalently

$$N'K^{-1}NR^{-1}N'-N'=0$$

$\Rightarrow\ N'K^{-1}(K-D)-N'=0$ **(Using** $D=K-NR^{-1}N'$**)**

$\Rightarrow\ N'K^{-1}D=0$ **(iii)**

Thus Q_i **and** P_j **are orthogonal if**

$$NR^{-1}N'K^{-1}N=N'$$

or equivalently $NR^{-1}C=0$

or equivalently $DK^{-1}N=0.$

Orthogonal block design:

A block design is said to be orthogonal if Q_i's and P_j's are orthogonal for all i and j. Thus the condition for the orthogonality of design is

$$NR^{-1}N'K^{-1}N = N,$$

$$NR^{-1}C = 0$$

or

$$DK^{-1}N = 0.$$

Orthogonal block design:

Lemma: If $\frac{n_{ij}}{r_j}$ is constant for all ***j***, then $\frac{n_{ij}}{k_i}$ is constant for all ***i*** and vice versa. In this case, we have $n_{ij} = \frac{k_i r_j}{n}$.

Proof: If $\frac{n_{ij}}{r_j}$ is constant for all j then $\frac{n_{ij}}{k_i} = q_i$, say.

$\Rightarrow n_{ij} = a_i r_j$

or $\sum_j n_{ij} = \sum_j a_i r_j = a_i \sum_j r_j = a_i n$

or $k_i = a_i n$ or $a_i = \frac{k_i}{n}$

Thus $\frac{n_{ij}}{r_j} = \frac{k_i}{n}$

or $n_{ij} = \frac{k_i r_j}{n}$

So $\frac{n_{ij}}{k_j} = \frac{r_j}{n}$: independent of i. Hence proved.

Contrast:

A linear function $\sum_{j=1}^{v} c_j \tau_j = C'\tau$ **where** $c_1, c_2, ..., c_v$ **are given number such that** $\sum_{j=1}^{v} c_j = 0$ **is called a contrast of** τ_j 's.

A contrast $\sum_{j=1}^{v} c_j \tau_j = C'\tau$ **with** $C = (c_1, c_2, ..., c_v)$ **in treatment effects** $\tau = (\tau_1, \tau_2, ..., \tau_v)$ **is called an elementary contrast if *C* has only two non-zero components 1 and -1.**

Elementary contrasts in the treatment effects involve all the differences in the form $\tau_i - \tau_j, \quad i \neq j.$

It is desirable to design experiments where all the elementary contrasts are estimable.

Connected Design:

A design where all the elementary contrasts are estimable is called a connected design otherwise it is called a disconnected design.

The physical meaning of connectedness of a design is as follows:

Given any two treatment effects τ_{i1} and τ_{i2}, it is possible to have a chain of treatment effects like $\tau_{i1}, \tau_{1j}, \tau_{2j}, \ldots, \tau_{nj}, \tau_{i2},$ such that two adjoining treatments in this chain occur in the same block.

In a connected design, within every block, all the treatment contrasts are estimable and pair-wise comparison of estimators have similar variances.

Example of connected design:

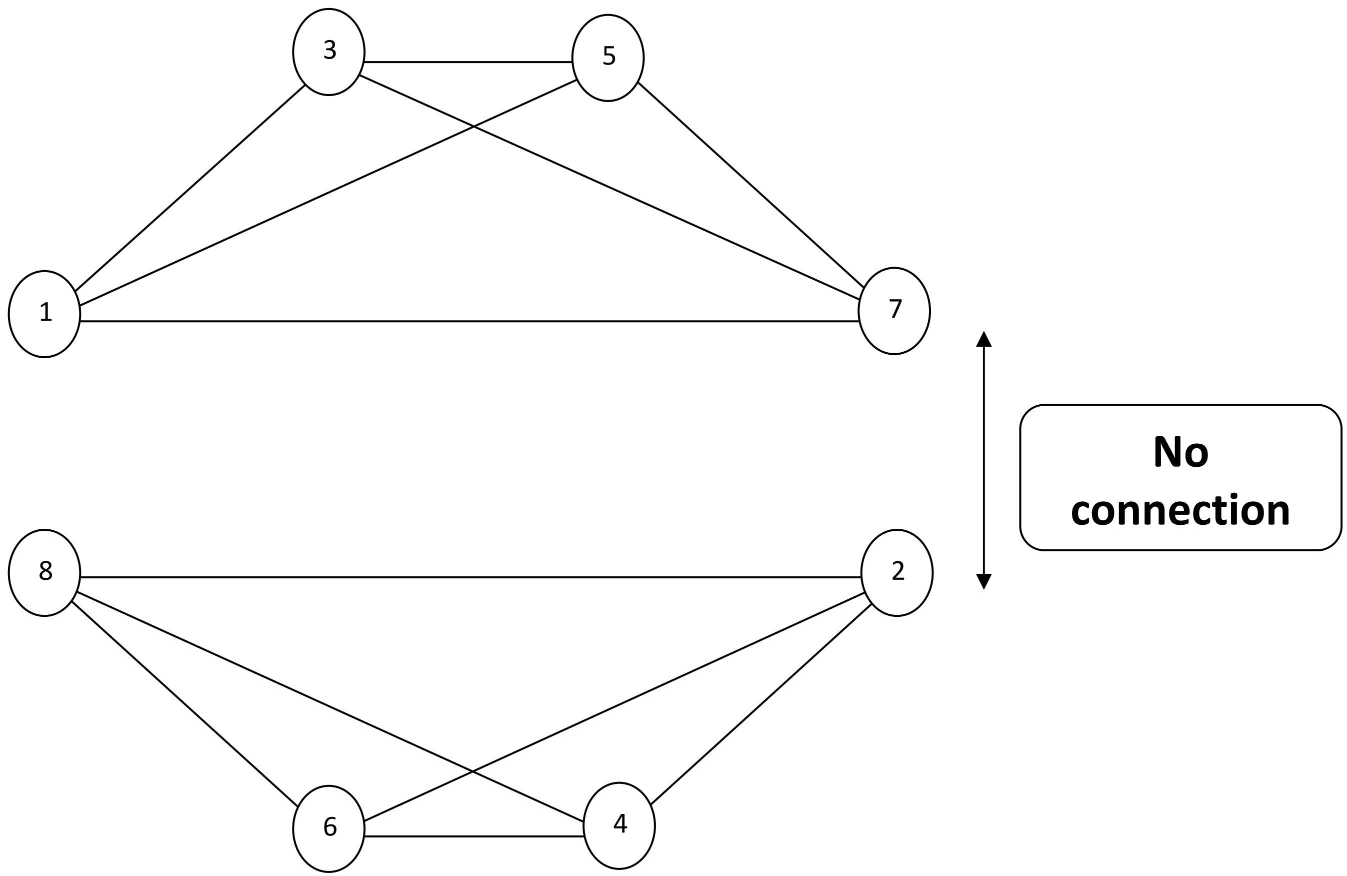
Consider a disconnected incomplete block design as follows:

b = 8 (Block numbers: I, II,…,VIII), k = 3, v = 8 (treatment numbers: 1,2,…,8), r = 3

Blocks	Treatments
I	1 3 5
II	2 4 6
III	3 5 7
IV	4 6 8
V	5 7 1
VII	6 8 2
VII	7 1 3
VIII	8 2 4

Example of connected design:

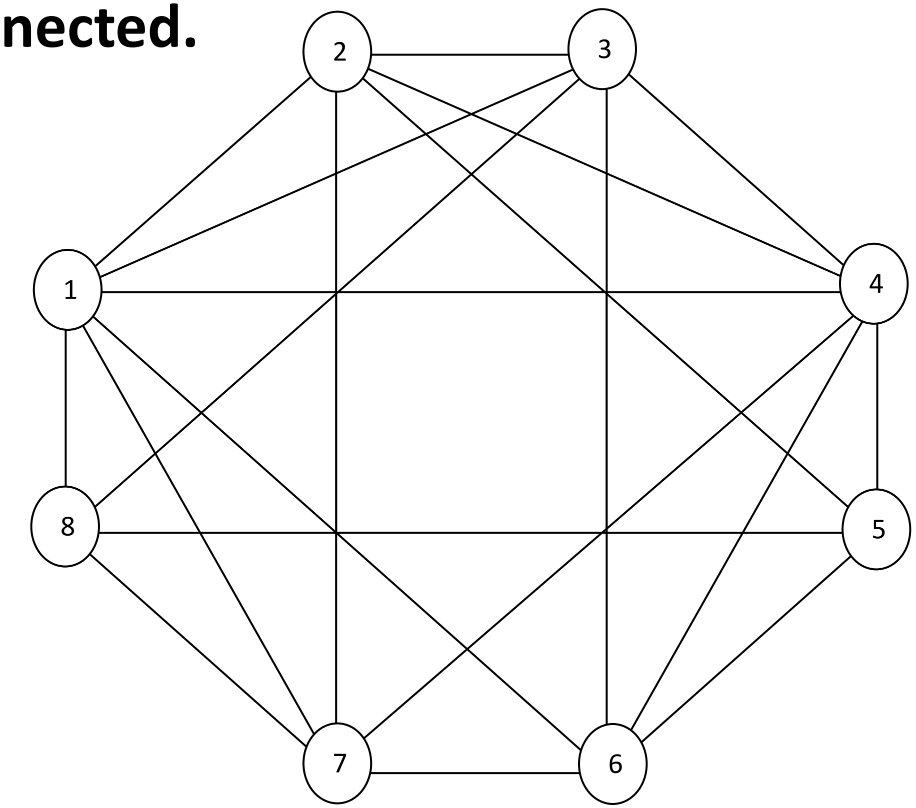
The blocks of this design can be represented graphically as follows:



Note that it is not possible to reach the treatment, e.g., 7 from 2, 3 from 4 etc. So the design is not connected.

Example of connected design:

Moreover, if the blocks of the design are given like in the following figure, then any treatment can be reached from any treatment. So the design, in this case, is connected.



Example of connected design:

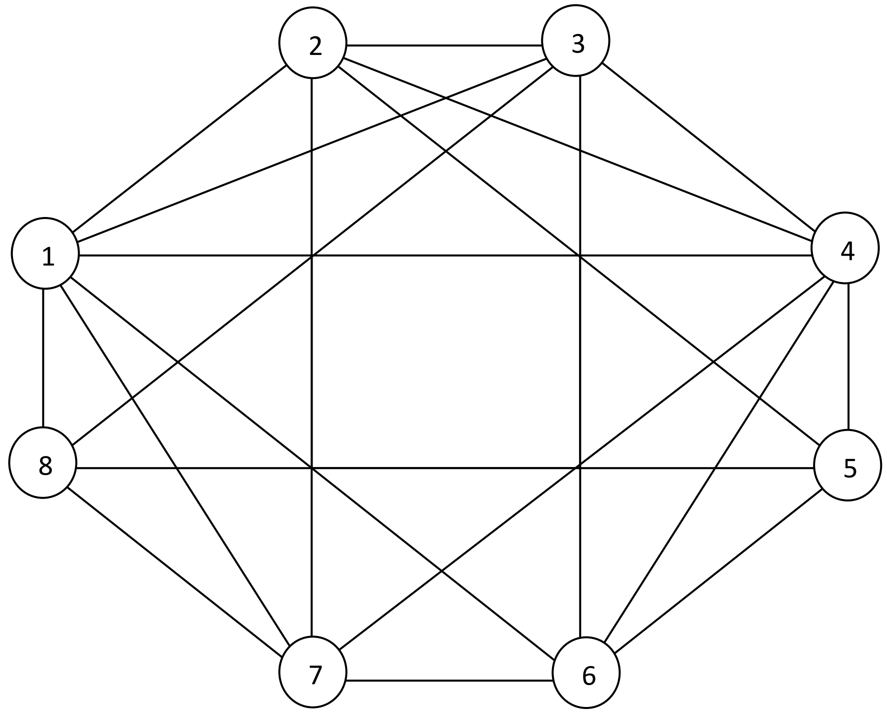
For example, treatment 2 can be reached from treatment 6 through different routes like

$6 \to 5 \to 4 \to 3 \to 2,$

or $6 \to 3 \to 2,$

or $6 \to 7 \to 8 \to 1 \to 2,$

or $6 \to 7 \to 2$



A design is connected if every treatment can be reached from every treatment via lines in the connectivity graph.

Properties of incomplete block designs:

Theorem: An incomplete block design with *v* treatments is connected if and only if $rank(C) = v - 1$.

Properties of incomplete block designs:

Lemma: **For a connected block design**

$Cov(Q,P)=0$ **if and only if** $N'=\frac{rk'}{n}.$

Proof: **"if" part**

When $N'=\frac{rk'}{n},$ **we have**

$$\frac{1}{\sigma^2}Cov(Q,P)=N'K^{-1}NR^{-1}N'-N'$$

$$=\frac{rk'K^{-1}kr'R^{-1}N'}{n^2}-N'$$

Properties of incomplete block designs:

Since $k'K^{-1}k = (k_1, k_2, \ldots, k_b) diag\left(\frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_b}\right) \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_b \end{pmatrix}$

$$= (1, \ldots, 1) \begin{pmatrix} k_1 \\ \vdots \\ k_b \end{pmatrix} = \sum_{i=1}^{b} k_i = n$$

and $r'R^{-1} = (r_1, r_2, \ldots, r_v) diag\left(\frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_v}\right) = E_{1v}.$

Then $\frac{1}{\sigma^2} Cov(Q, P) = \frac{rnE_{1v}N'}{n^2} - N'$

$$= \frac{rE_{1v}N'}{n} - N' = \frac{rk'}{n} - N' \quad = N' - N' = 0.$$

Properties of incomplete block designs:

"Only if part"

Let $Cov(Q,P)=0$

$\Rightarrow N'K^{-1}NR^{-1}N'-N'=0$ (**Since** $C=R-N'K^{-1}N$)

or $(R-C)R^{-1}N'-N'=0$

or $CR^{-1}N'=0.$

Let $R^{-1}N'=A=(\underline{a}_1,\underline{a}_2,...,\underline{a}_b)$ **where** $\underline{a}_1,\underline{a}_2,...,\underline{a}_b$ **are the columns of** A.

Since the design is connected, so the columns of *A* are proportional to E_{v1}**. Also, all row/column sums of C are zero.**

So $(CE_{v1},CE_{v1},...,CE_{v1})=0$ **and** $CA=0$

or $C(\underline{a}_1,\underline{a}_2,...,\underline{a}_b)=0 \quad \Rightarrow a_i \propto E_{v1}$

or $a_i=\alpha_i E_{v1}\ ;\ i=1,2,...,b$ **where** α_i**'s are some scalars.**

Properties of incomplete block designs:

This gives $A = R^{-1}N' = E_{v1}\alpha'$ **where** $\alpha = (\alpha_1, \alpha_2, ..., \alpha_b)'$.

So we have $N' = RE_{v1}\alpha = (r_1, r_2, .., r_v)'\alpha' = r\alpha'$ **where** $r = (r_1, r_2, .., r_v)'$.

Pre-multiply by E_{1v} **gives**

$$E_{1v}N' = (k_1, k_2, ..., k_b)' = E_{1v}r\alpha' = n\alpha'$$

or $k = n\alpha'$

$$\Rightarrow \alpha' = \frac{k'}{n}$$ **where** $k = (k_1, k_2, ..., k_b)'$

Thus

$$N' = r\alpha' = \frac{rk'}{n}.$$ **Hence proved.**

Properties of incomplete block designs:

A connected block design is said to be orthogonal if and only if the incidence matrix of the design satisfies the condition $N' = \frac{rk'}{n}$.

Designs which do not satisfy this condition are called non-orthogonal. It is clear from this result that if at least one entry of *N* is zero, the design cannot be orthogonal.

A block design with at least one zero-entry in its incidence matrix is called an incomplete block design.

Properties of incomplete block designs:

Theorem: A sufficient condition for the orthogonality of design is that $\frac{n_{ij}}{r_j}$ is constant for all *j*.

Conclusion: It is obvious from the condition of orthogonality of a design that a design which is not connected and an incomplete design even though it may be connected cannot have an orthogonal structure.

Randomized block design:

Now we illustrate the general nature of the incomplete block design. We try to obtain the results for a randomized block design through the results of an incomplete block design.

The randomized block design is an arrangement of *v* treatment in *b* blocks of *v* plots each, such that every treatment occurs in every block, one treatment in each plot.

The arrangement of treatment within a block is random and in terms of incidence matrix, $n_{ij} = 1$ **for all** $i = 1, 2, \ldots, b; \ \ j = 1, 2, \ldots, v.$

Thus we have $k_i = \sum_j n_{ij} = v$ **for all** $i,$ $\ \ r_j = \sum_j n_{ij} = b$ **for all** $j.$

Randomized block design:

We have $\frac{n_{ij}}{r_j} = \frac{1}{b}$ constant for all j.

$$C_{jj} = b - \frac{b}{v} \quad , \quad C_{jj'} = -\frac{b}{v} \quad , \quad Q_j = V_j - \frac{G}{v}.$$

Normal equations for $\tau's$ are

$$\left(b - \frac{b}{v}\right)\tau_j - \frac{b}{v}\sum_{i \neq j'=1}^{v} \tau_{j'} = V_j - \frac{G}{v}; \quad j = 1....v$$

$$\tau_1 + \tau_2 .... + \tau_v = 0.$$

Thus $\quad b\tau_j - \frac{b}{v}\sum_j \tau_j = V_j - \frac{G}{v}$

or $\quad \hat{\tau}_j = \frac{1}{b}\left(V_j - \frac{G}{v}\right) = \bar{y}_{oj} - \bar{y}_{oo}.$

Randomized block design:

The sum of squares due to treatments adjusted for blocks is

$$= \sum_j \hat{\tau}_j Q_j$$

$$= \frac{1}{b} \sum_j \left(V_j - \frac{G}{v} \right)^2$$

$$= \frac{\sum_j V_j^2}{b} - \frac{G^2}{bv},$$

which is also the sum of squares due to treatments which are unadjusted for blocks because the design is orthogonal.

Randomized block design:

Sum of squares due to blocks $= \frac{\sum_i B_i^2}{v} - \frac{G^2}{bv}$

Sum of squares due to error $= \sum_i \sum_j \left(y_{ij} - \frac{B_i}{v} - \frac{V_j}{b} + \frac{G}{bv} \right)^2.$

These expressions are the same as obtained under the analysis of variance in the set up of a randomized block design.