Analysis of Variance and Design of Experiments

Incomplete Block Designs and Their Analysis

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Lecture 29

Recovery of Interblock Information in IBD

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Estimates of linear contrast of μ* and τ:

We now have two different estimates of the treatment effect as

- **based on intrablock analysis** $\hat{\tau} = C^{-}Q$ **and**
- **based on interblock analysis** $\tilde{\tau} = (N'N)^{-1}N'B - \frac{GE_{v1}}{bk}$.

Let us consider the estimation of linear contrast of treatment effects $L = l'\tau$ **.**

Since the intrablock and interblock estimates of τ **are based on Gauss-Markov model and least-squares principle, so the best estimate of** L **based on intrablock estimation is obtained by replacing the respective estimators.**

Estimates of linear contrast of μ* and τ:

Since the intrablock and interblock estimates of τ are based on Gauss-Markov model and least-squares principle, so the best estimate of L based on intrablock estimation is

$$L_1 = l'\hat{\tau} = l'C^{-}Q$$

and the best estimate of L based on interblock estimation is

$$L_2 = l'\tilde{\tau}$$

$$= l'\left[(N'N)^{-1} N'B - \frac{GE_{v1}}{bk} \right]$$

$$= l'(N'N)^{-1} N'B \quad (\text{since } l'E_{v1} = 0 \text{ being contrast.})$$

Variance and covariance of estimates of linear contrasts:

The variances of L_1 and L_2 are

$$Var(L_1) = \sigma^2 l' C^- l$$

and

$$Var(L_2) = \sigma_f^2 l'(N'N)^{-1} l,$$

respectively. The covariance between *Q* (from intrablock) and *B* (from interblock) is

$$\begin{aligned} Cov(Q, B) &= Cov(V - N'K^{-1}B^*, B) \\ &= Cov(V, B) - Cov(N'K^{-1}B^*, B) \\ &= N'\sigma_f^2 - N'K^{-1}K\sigma_f^2 \\ &= 0. \end{aligned}$$

Note that *B denotes the block total based on intrablock analysis and *B* denotes the block totals based on interblock analysis.**

Variance and covariance of estimates of linear contrasts:

We are using two notations *B* and *B just to indicate that the two block totals are different. The reader should not misunderstand that it follows from the result $Cov(Q, B) = 0$ in case of intrablock analysis.**

Thus

$$Cov(L_1, L_2) = 0$$

irrespective of the values of *l*.

The question now arises that given the two estimators $\hat{\tau}$ and $\tilde{\tau}$ of τ, how to combine them and obtain a minimum variance unbiased estimator of τ.

Construction of estimators: Example

It is illustrated with the following example:

Let $\hat{\varphi}_1$ **and** $\hat{\varphi}_2$ **be any two unbiased estimators of a parameter** φ **with** $Var(\hat{\varphi}_1) = \sigma_1^2$ **and** $Var(\hat{\varphi}_2) = \sigma_2^2$**.**

Consider a linear combination $\hat{\varphi} = \theta_1 \hat{\varphi}_1 + \theta_2 \hat{\varphi}_2$ **with weights** θ_1 **and** θ_2**.**

In order that $\hat{\varphi}$ **is an unbiased estimator of** φ**, we need**

$$E(\hat{\varphi}) = \varphi$$

or $$\theta_1 E(\hat{\varphi}_1) + \theta_2 E(\hat{\varphi}_2) = \varphi$$

or $$\theta_1 \varphi + \theta_2 \varphi = \varphi$$

or $$\theta_1 + \theta_2 = 1.$$

Construction of estimators: Example

So modify $\hat{\varphi}$ **as** $\dfrac{\theta_1\hat{\varphi}_1 + \theta_2\hat{\varphi}_2}{\theta_1 + \theta_2}$ **which is the weighted mean of** $\hat{\varphi}_1$ **and** $\hat{\varphi}_2$**.**

Further, if $\hat{\varphi}_1$ **and** $\hat{\varphi}_2$ **are independent, then**

$$Var(\hat{\varphi}) = \theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2.$$

Now we find θ_1 **and** θ_2 **such that** $Var(\hat{\varphi})$ **is minimum such that** $\theta_1 + \theta_2 = 1$**.**

$$\frac{\partial Var(\hat{\varphi})}{\partial \theta_1} = 0 \Rightarrow 2\theta_1\sigma_1^2 - 2(1-\theta_1)\sigma_2^2 = 0$$

or $\theta_1\sigma_1^2 - \theta_2\sigma_2^2 = 0$

or $\dfrac{\theta_1}{\theta_2} = \dfrac{\sigma_2^2}{\sigma_1^2}$

or **weight** $\propto \dfrac{1}{\textbf{variance}}.$

Construction of estimators: Example

Alternatively, the Lagrangian function approach can be used to obtain such a result as follows. The Lagrangian function with λ^* as Lagrangian multiplier is given by

$$\phi = Var(\hat{\varphi}) - \lambda^*(\theta_1 + \theta_2 - 1)$$

and solving

$$\frac{\partial \phi}{\partial \theta_1} = 0, \frac{\partial \phi}{\partial \theta_2} \text{ and } \frac{\partial \phi}{\partial \lambda^*} = 0$$

also gives the same result that

$$\frac{\theta_1}{\theta_2} = \frac{\sigma_2^2}{\sigma_1^2}.$$

Construction of estimators:

We note that a pooled estimator of τ in the form of weighted arithmetic mean of uncorrelated L_1 and L_2 is the minimum variance unbiased estimator of τ when the weights θ_1 and θ_2 of L_1 and L_2, respectively are chosen such that $\dfrac{\theta_1}{\theta_2} = \dfrac{Var(L_2)}{Var(L_1)}$,

i.e., the chosen weights are reciprocal to the variance of respective estimators, irrespective of the values of *l*. So consider the weighted average of L_1 and L_2 with weights θ_1 and θ_2, respectively as

$$\tau^* = \frac{\theta_1 L_1 + \theta_2 L_2}{\theta_1 + \theta_2} = \frac{l'(\theta_1 \hat{\tau} + \theta_2 \tilde{\tau})}{\theta_1 + \theta_2}$$

with

$$\theta_1^{-1} = l'C^{-}l\sigma^2$$

$$\theta_2^{-1} = l'(N'N)^{-1} l\sigma_f^2 .$$

Construction of estimators:

The linear contrast of τ^* is

$$L^* = l'\tau^*$$

and its variance is

$$Var(L^*) = \frac{\theta_1^2 Var(L_1) + \theta_2^2 Var(L_2)}{(\theta_1 + \theta_2)^2} l'l \quad \text{(since } Cov(L_1, L_2) = 0\text{)}$$

$$= \frac{l'l}{(\theta_1 + \theta_2)}$$

because the weights of estimators are chosen to be inversely proportional to the variance of the respective estimators. We note that τ^* can be obtained provided θ_1 and θ_2 are known.

Note that the constant of proportionality gets cancelled out.

Construction of estimators:

But θ_1 and θ_2 are known only if σ^2 and σ_β^2 are known.

So τ^* can be obtained when σ^2 and σ_β^2 are known.

In case, if σ^2 and σ_β^2 are unknown, then their estimates can be used.

A question arises how to obtain such estimators?

Construction of estimators:

One such approach to obtain the estimates of σ^2 and σ_β^2 is based on utilizing the results from intrablock and interblock analysis both and is as follows.

From intrablock analysis

$$E(SS_{Error(t)}) = (n - b - v + 1)\sigma^2,$$

so an unbiased estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{SS_{Error(t)}}{n - b - v + 1}.$$

An unbiased estimator of τ is obtained by using the following results based on the intrablock analysis:

Construction of estimators:

An unbiased estimator of σ_β^2 is obtained by using the following results based on the intrablock analysis:

$$SS_{Treat(unadj)} = \sum_{j=1}^{v} \frac{V_j^2}{\tau_j} - \frac{G^2}{n}, \qquad SS_{Block(unadj)} = \sum_{i=1}^{b} \frac{B_i^2}{k_i} - \frac{G^2}{n},$$

$$SS_{Treat(adj)} = \sum_{j=1}^{v} Q_j \hat{\tau}_j, \qquad SS_{Total} = \sum_{i=1}^{b} \sum_{j=1}^{v} y_{ij}^2 - \frac{G^2}{n},$$

where

$$SS_{Total} = SS_{Treat(adj)} + SS_{Block(unadj)} + SS_{Error(t)} = SS_{Treat(unadj)} + SS_{Block(adj)} + SS_{Error(t)}.$$

Hence

$$SS_{Block(adj)} = SS_{Treat(adj)} + SS_{Block(unadj)} - SS_{Treat(unadj)}.$$

Construction of estimators:

Under the interblock analysis model

$$E[SS_{Block(adj)}] = E[SS_{Treat(adj)}] + E[SS_{Block(unadj)}] - E[SS_{Treat(unadj)}]$$

which is obtained as follows:

$$E[SS_{Block(adj)}] = (b-1)\sigma^2 + (n-v)\sigma_\beta^2$$

or

$$E\left[SS_{Block(adj)} - \frac{b-1}{n-b-v+1} SS_{Error(t)} \right] = (n-v)\sigma_\beta^2.$$

Thus an unbiased estimator of σ_β^2 is

$$\hat{\sigma}_\beta^2 = \frac{1}{n-v}\left[SS_{Block(adj)} - \frac{b-1}{n-b-v+1} SS_{Error(t)} \right].$$

Construction of estimators:

Now the estimates of weights θ_1 and θ_2 can be obtained by replacing σ^2 and σ_β^2 by $\hat{\sigma}^2$ and $\hat{\sigma}_\beta^2$ respectively.

Then the estimate of θ_1 and θ_2 can be obtained by replacing by their estimates and can be used in place of τ^*.

It may be noted that the exact distribution of the associated sum of squares due to treatments is difficult to find when σ^2 and σ_β^2 are replaced by $\hat{\sigma}^2$ and $\hat{\sigma}_\beta^2$, respectively in τ^*.

Construction of estimators:

Some approximate results are possible which we will present while dealing with the balanced incomplete block design.

An increase in the precision using interblock analysis as compared to intrablock analysis is measured by

$$\frac{\text{1/variance of pooled estimate}}{\text{1/variance of intrablock estimate}} - 1.$$

Construction of estimators:

In the interblock analysis, the block effects are treated as a random variable which is appropriate if the blocks can be regarded as a random sample from a large population of blocks.

The best estimate of the treatment effect from the intrablock analysis is further improved by utilizing the information on block totals.

Since the treatments in different blocks are not all the same, so the difference between block totals is expected to provide some information about the differences between the treatments.

Construction of estimators:

So the interblock estimates are obtained and pooled with intrablock estimates to obtain the combined estimate of τ.

The procedure of obtaining the interblock estimates and then obtaining the pooled estimates is called the recovery of interblock information.