Analysis of Variance and Design of Experiments

Balanced Incomplete Block Design

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Lecture 31

Basic Definitions and Intrablock Analysis of Variance in BIBD

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Balanced Incomplete Block Design:

Five parameters denote such design as $D(b,k,v,r;\lambda)$**.**

The parameters b,k,v,r **and** λ **are not chosen arbitrarily.**

They satisfy the following relations:

$(I)\quad bk = vr$

$(II)\quad \lambda(v-1) = r(k-1)$

(III) $b \geq v$ (**and hence** $r > k$).

Interpretation of conditions of BIBD: (i) $bk = vr$

This condition is related to the total number of plots in an experiment.

In our settings, there are k plots in each block and there are b blocks.

So the total number of plots are bk.

Further, there are v treatments and each treatment is replicated r times such that each treatment occurs at most in one block.

So total number of plots containing all the treatments is vr.

Since both the statements counts the total number of plots, hence $bk = vr$.

Interpretation of conditions of BIBD: (ii) $r(k - 1) = \lambda(v - 1)$

Each block has *k* plots.

Thus the total pairs of plots in a block = $\binom{k}{2} = \frac{k(k-1)}{2}.$

There are *b* blocks. Thus the total pairs of plots such that each pair consists of plots within a block = $b\frac{k(k-1)}{2}$ **.**

There are *v* treatments, thus the total number of pairs of treatment = $\binom{v}{2} = \frac{v(v-1)}{2}$

Each pair of treatment is replicated λ **times, i.e., each pair of treatment occurs in** λ **blocks.**

Interpretation of conditions of BIBD: (ii) $r(k - 1) = \lambda(v - 1)$

Thus the total number of pairs of plots within blocks must be

$$= \lambda \frac{v(v-1)}{2} \; .$$

Hence $$b\frac{k(k-1)}{2} = \lambda \frac{v(v-1)}{2}$$

Using $bk = vr$ **in this relation, we get** $r\left(k-1\right) = \lambda\left(v-1\right).$

Proof of (III) was given by Fisher but quite long, so not needed here.

Balancing in designs:

There are two types of balancing – Variance balanced and efficiency balanced. We discuss the variance balancing now and the efficiency balancing later.

Balanced Design (Variance Balanced):

A connected design is said to be balanced (variance balanced) if all the elementary contrasts of the treatment effects can be estimated with the same precision.

This definition does not hold for the disconnected design, as all the elementary contrasts are not estimable in this design.

Proper designs:

An incomplete block design with $k_1 = k_2 = ... = k_b = k$ is called a proper design.

Symmetric BIBD:

A BIBD is called symmetrical if number of blocks = number of treatments i.e. ***b = v***

Since ***b = v*** so from ***bk = vr***

$$\Rightarrow k = r.$$

Thus the number of pairs of treatments common between any two blocks = λ .

Resolvable design:

A block design of

- *b* blocks in which
- each of *v* treatments is replicated *r* times

is said to be resolvable if *b* blocks can be divided into *r* sets of *b*/*r* blocks each, such that every treatment appears in each set precisely once.

Obviously, in a resolvable design, *b* is a multiple of *r*.

Resolvable design:

Theorem: **If in a BIBD** $D(v,b,r,k,\lambda), b$ **is divisible by *r*, then**

$$b \geq v + r - 1.$$

Proof: **Let *b = n r* where (*n* > 1 is a positive integer).**

For a BIBD, $\lambda(v-1) = r(k-1)$

$$\text{or} \quad r = \frac{\lambda(v-1)}{(k-1)} \qquad \begin{bmatrix} \text{because } vr = bk \\ \text{or} \quad vr = nrk \\ \text{or} \quad v = nk \end{bmatrix}$$

$$= \frac{\lambda(nk-1)}{(k-1)} = \lambda\left(\frac{n-1}{k-1}\right) + \lambda n.$$

Since $n > 1$ **and** $k > 1$, **so** $\lambda n > 1$ **is an integer. Since** r **has to be an integer.**

$$\Rightarrow \lambda \frac{(n-1)}{k-1} \text{ is also a positive integer.}$$

Resolvable design:

Now, if possible, let

$$b < v + r - 1$$

$$\Rightarrow nr < v + r - 1$$

or $r(n-1) < v - 1$

or $r(n-1) < \dfrac{r(k-1)}{\lambda}$ **(because** $v - 1 = \dfrac{r(k-1)}{\lambda}$**)**

$$\Rightarrow \frac{\lambda(n-1)}{k-1} < 1$$

which is a contradiction as integer can not be less than one

$\Rightarrow b < v + r - 1$ **is impossible. Thus the opposite is true.**

$\Rightarrow b \geq v + r - 1$ **holds correct.**

Intrablock analysis of BIBD:

Consider the model

$$y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij}; \; i = 1, 2, ..., b; \; j = 1, 2, ..., v,$$

where μ **is the general mean effect**;

β_i **is the fixed additive ith block effect**;

τ_j **is the fixed additive jth treatment effect and**

ε_{ij} **is the i.i.d. random error with** $\varepsilon_{ij} \sim N(0, \sigma^2)$.

We don't need to develop the analysis of BIBD from starting.

Since BIBD is also an incomplete block design and the analysis of incomplete block design has already been presented in the earlier module, so we implement those derived expressions directly under the setup and conditions of BIBD.

Intrablock analysis of BIBD:

Using the same notations, we represent the

- **blocks totals by** $B_i = \sum_{j=1}^{v} y_{ij}$ **,**
- **treatment totals by** $V_j = \sum_{i=1}^{b} y_{ij}$**,**
- **adjusted treatment totals by** Q_j **and**
- **grand total by** $G = \sum_i \sum_j y_{ij}$.

Intrablock analysis of BIBD:

Then the block effects are eliminated from the normal equations and the normal equations are solved for the treatment effects.

The resulting intrablock equations of treatment effects in matrix notations are expressible as $Q = C\hat{\tau}$ **.**

Now we obtain the forms of *C* and *Q* in the case of BIBD.

Intrablock analysis of BIBD:

The diagonal elements of *C* are given by

$$c_{jj} = r - \frac{\sum_{i=1}^{b} n_{ij}^2}{k} \quad (j = 1, 2, ..., v)$$

$$= r - \frac{r}{k},$$

The off-diagonal elements of *C* are given by

$$c_{jj'} = -\frac{1}{k} \sum_{i=1}^{b} n_{ij} n_{ij'} \quad (j \neq j'; j, j' = 1, 2, ..., v)$$

$$= -\frac{\lambda}{k}.$$

Intrablock analysis of BIBD:

The adjusted treatment totals are obtained as

$$Q_j = V_j - \frac{1}{k}\sum_{i=1}^{b} n_{ij} B_i \quad (j \neq 1, 2, ..., v)$$

$$= V_j - \frac{1}{k}\sum_{i(j)} B_i$$

where $\sum_{i(j)}$ **denotes the sum over those blocks containing** ***j*** th **treatment.**

Denote $T_j = \sum_{i(j)} B_i,$

then

$$Q_j = V_j - \frac{T_j}{k}.$$

Intrablock analysis of BIBD:

The *C* matrix is simplified as follows:

$$
\begin{aligned}
C &= rI - \frac{N'N}{k} \\
&= rI - \frac{1}{k}\left[(r-\lambda)I + \lambda E_{v1}E_{v1}^{'}\right] \\
&= r\left(\frac{k-1}{k}\right)I + \frac{\lambda}{k}(I - E_{v1}E_{v1}^{'}) \\
&= \lambda\left(\frac{v-1}{k}\right)I + \frac{\lambda}{k}(I - E_{v1}E_{v1}^{'}) \\
&= \frac{\lambda v}{k}\left(I - \frac{E_{v1}E_{v1}^{'}}{v}\right).
\end{aligned}
$$

Intrablock analysis of BIBD:

Since *C* is not as a full rank matrix, so its unique inverse does not exist. The generalized inverse of *C* is denoted as C^- **which is obtained as**

$$C^- = \left(C + \frac{E_{v1}E_{v1}^{'}}{v} \right)^{-1}.$$

Since

$$C = \frac{\lambda v}{k}\left(I_v - \frac{E_{v1}E_{v1}^{'}}{v} \right)$$

or $$\frac{kC}{\lambda v} = I_v - \frac{E_{v1}E_{v1}^{'}}{v},$$

the generalized inverse of $\frac{k}{\lambda v}C$ **is obtained as follows:**

Intrablock analysis of BIBD:

$$\left(\frac{k}{\lambda v}\right)^{-1} C^{-} = \left[C + \frac{E_{v1}E_{v1}^{'}}{v},\right]^{-1}$$

$$= \left[I_v - \frac{E_{v1}E_{v1}^{'}}{v} + \frac{E_{v1}E_{v1}^{'}}{v}\right]^{-1}$$

$$= I_v.$$

Thus $C^{-} = \frac{\lambda v}{k} I_v$ **.**

Thus an estimate of τ **is obtained from** $Q = C\tau$ **as**

$$\hat{\tau} = C^{-}Q$$

$$= \frac{\lambda v}{k} Q.$$