Analysis of Variance and Design of Experiments

Balanced Incomplete Block Design

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Lecture 32

Intrablock Analysis of Variance and Other Tests in BIBD

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Intrablock analysis of BIBD:

Consider the model

$$y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij};\ i = 1, 2, ..., b;\ j = 1, 2, ..., v,$$

where μ **is the general mean effect;**

β_i **is the fixed additive** i^{th} **block effect;**

τ_j **is the fixed additive** j^{th} **treatment effect and**

ε_{ij} **is the i.i.d. random error with** $\varepsilon_{ij} \sim N(0, \sigma^2)$.

Intrablock analysis of BIBD:

Using the same notations, we represent the

- **blocks totals by** $B_i = \sum_{j=1}^{v} y_{ij}$ **,**
- **treatment totals by** $V_j = \sum_{i=1}^{b} y_{ij}$ **,**
- **adjusted treatment totals by** Q_j **and**
- **grand total by** $G = \sum_i \sum_j y_{ij}.$

Intrablock analysis of BIBD:

Then the block effects are eliminated from the normal equations and the normal equations are solved for the treatment effects.

The resulting intrablock equations of treatment effects in matrix notations are expressible as $Q = C\hat{\tau}$ **.**

Intrablock analysis of BIBD:

The diagonal elements of C **are given by**

$$c_{jj} = r - \frac{r}{k} \quad (j = 1, 2, ..., v)$$

The off-diagonal elements of C **are given by**

$$c_{jj'} = -\frac{\lambda}{k} \quad (j \neq j'; j, j' = 1, 2, ..., v)$$

The adjusted treatment totals are obtained as

$$Q_j = V_j - \frac{T_j}{k}.$$

Where $T_j = \sum_{i(j)} B_i,$ **and** $\sum_{i(j)}$ **denotes the sum over those blocks containing** j^{th} **treatment.**

Intrablock analysis of BIBD:

The C matrix in case of BIBD is as follows:

$$C = \frac{\lambda V}{k}\left(I - \frac{E_{v1}E_{v1}^{'}}{v} \right).$$

The generalized inverse of C matrix is

$$C^{-} = \frac{\lambda v}{k} I_v$$

Thus an estimate of τ is obtained from $Q = C\tau$ as

$$\hat{\tau} = C^{-}Q$$

$$= \frac{\lambda v}{k} Q.$$

Intrablock analysis of BIBD:

The null hypothesis of our interest is

$$H_0 : \tau_1 = \tau_2 = \ldots = \tau_v$$

against the alternative hypothesis

$H_1 :$ **at least one pair of** τ_j **'s is different.**

Now we obtain the various sum of squares involved in the development of analysis of variance as follows.

The adjusted treatment sum of squares is

$$SS_{Treat(adj)} = \hat{\tau}'Q = \frac{k}{\lambda v} Q'Q = \frac{k}{\lambda v} \sum_{j=1}^{v} Q_j^2 ,$$

Intrablock analysis of BIBD:

The unadjusted block sum of squares is

$$SS_{Block(unadj)} = \sum_{i=1}^{b} \frac{B_i^2}{k} - \frac{G^2}{bk}.$$

The total sum of squares is

$$SS_{Total} = \sum_{i=1}^{b} \sum_{j=1}^{v} y_{ij}^2 - \frac{G^2}{bk}.$$

The residual sum of squares is obtained by

$$SS_{Error(t)} = SS_{Total} - SS_{Block(unadj)} - SS_{Treat(adj)}.$$

Intrablock analysis of BIBD:

A test for $H_0 : \tau_1 = \tau_2 = ... = \tau_v$ is then based on the statistic

$$F_{Tr} = \frac{SS_{Treat(adj)} / (v-1)}{SS_{Error(t)} / (bk-b-v+1)}$$

$$= \frac{k}{\lambda v} . \frac{bk-b-v+1}{v-1} . \frac{\sum_{j=1}^{v} Q_j^2}{SS_{Error(t)}}$$

If $F_{Tr} > F_{1-\alpha, v-1, bk-b-v+1;}$ then H_0 is rejected.

This completes the analysis of variance test and is termed as intrablock analysis of variance.

This analysis can be compiled into the intrablock analysis of variance table for testing the significance of the treatment effect given as follows.

Intrablock analysis of BIBD:

Intrablock analysis of variance table of BIBD for $H_0 : \tau_1 = \tau_2 = ... = \tau_v$

Source	Sum of squares	Degrees of freedom	Mean squares	*F*
Between treatment (adjusted)	$SS_{Treat(adj)}$	v - 1	$MS_{treat} = \frac{SS_{Treat(adj)}}{v-1}$	$\frac{MS_{Treat}}{MS_E}$
Between blocks (unadjusted)	$SS_{Block(unadj)}$	b - 1		
Intrablock error	$SS_{Error(t)}$ (by substraction)	$bk - b - v + 1$	$MS_E = \frac{SS_{Error(t)}}{bk-b-v+1}$	
Total	$SS_{Total} = \sum_i \sum_j y_{ij}^2 - \frac{G^2}{bk}$	$bk - 1$		

Intrablock analysis of BIBD:

In case, the null hypothesis is rejected, then we go for a pairwise comparison of the treatments.

For that, we need an expression for the variance of the difference of two estimates of treatment effects.

So we obtain the variance of an elementary contrast $(\hat{\tau}_j - \hat{\tau}_{j'},\ j \neq j')$ **under the intrablock analysis .**

Intrablock analysis of BIBD:

The variance of an elementary contrast $(\hat{\tau}_j - \hat{\tau}_{j'}, j \neq j')$ under the intrablock analysis .

$$V^* = Var(\hat{\tau}_j - \hat{\tau}_{j'}) = Var\left(\frac{k}{\lambda v}(Q_j - Q_{j'})\right)$$

$$= \frac{k^2}{\lambda^2 v^2}[Var(Q_j) + Var(Q_{j'}) - 2Cov(Q_j Q_{j'})]$$

$$= \frac{k^2}{\lambda^2 v^2}(c_{jj} + c_{j'j'} - 2c_{jj'})\sigma^2$$

$$= \frac{k^2}{\lambda^2 v^2}\left[2r\left(1 - \frac{1}{k}\right) + \frac{2\lambda}{k}\right]\sigma^2 = \frac{2k}{\lambda v}\sigma^2.$$

This expression depends on σ^2 which is unknown. So it is unfit for use in the real data applications. One solution is to estimate σ^2 from the given data and use it in the place of σ^2 .

Intrablock analysis of BIBD:

An unbiased estimator of σ^2 **is** $\hat{\sigma}^2 = \dfrac{SS_{Error(t)}}{bk - b - v + 1}.$

Thus an unbiased estimator of *V can be obtained by substituting** $\hat{\sigma}^2$ **in it as**

$$\hat{V}_* = \frac{2k}{\lambda v} . \frac{SS_{Error(t)}}{bk - b - v + 1}.$$

If H_0 **is rejected, then we make pairwise comparison and use the multiple comparison test.**

Intrablock analysis of BIBD:

To test $H_0 : \tau_j = \tau_{j'} (j \neq j')$, **a suitable statistic is**

$$t = \frac{k(bk - b - v + 1)}{\lambda v} \cdot \frac{Q_j - Q_{j'}}{\sqrt{SS_{Error(t)}}}$$

which follows a *t*-distribution with (*bk* – *b* – *v* + 1) degrees of freedom under H_0.

Comparison of RBD with BIBD:

A question arises that how a BIBD compares to an RBD. Note that BIBD is an incomplete block design whereas RBD is a complete block design. This point should be kept in mind while making such restrictive comparison.

We now compare the efficiency of BIBD with a randomized block (complete) design with *r* replicates. The variance of an elementary contrast under a randomized block design (RBD) is

$$V_R^* = Var(\hat{\tau}_j - \hat{\tau}_{j'})_{RBD} = \frac{2\sigma_*^2}{r}$$

where $Var(y_{ij}) = \sigma_*^2$ **under RBD.**

Comparison of RBD with BIBD:

Thus the relative efficiency of BIBD relative to RBD is

$$\frac{Var(\hat{\tau}_j - \hat{\tau}_{j'})_{RBD}}{Var(\hat{\tau}_j - \hat{\tau}_{j'})} = \frac{\left(\dfrac{2\sigma_*^2}{r}\right)}{\left(\dfrac{2k\sigma^2}{\lambda v}\right)} = \frac{\lambda v}{rk}\left(\frac{\sigma_*^2}{\sigma^2}\right).$$

The factor $\dfrac{\lambda v}{rk} = E$ **(say) is termed as the efficiency factor of BIBD and**

$$E = \frac{\lambda v}{rk} = \frac{v}{k}\left(\frac{k-1}{v-1}\right) = \left(1-\frac{1}{k}\right)\left(1-\frac{1}{v}\right)^{-1} < 1 \text{ (since } v > k).$$

The actual efficiency of BIBD over RBD not only depends on the efficiency factor but also on the ratio of variances σ_*^2/σ^2**. So BIBD can be more efficient than RBD as** σ_*^2 **can be more than** σ^2 **because** $k < v.$

Efficiency balanced design:

A block design is said to be efficiency balanced if every contrast of the treatment effects is estimated through the design with the same efficiency factor.

If a block design satisfies any two of the following properties:

i. efficiency balanced,

ii. variance balanced and

iii. an equal number of replications,

then the third property also holds true.

Missing observations in BIBD:

The intrablock estimate of missing $(i, j)^{th}$ observation y_{ij} is

$$y_{ij} = \frac{vr(k-1)B_i - k(v-1)Q_j - (v-1)Q_j^{'}}{k(k-1)(bk-b-v+1)}$$

$Q_j^{'}$: **the sum of Q value for all other treatment (but not the j^{th} one) which are present in the i^{th} block.**

All other procedures remain the same.