Analysis of Variance and Design of Experiments

Balanced Incomplete Block Design

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Lecture 33

Recovery of Interblock Information

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Interblock analysis and recovery of interblock information in BIBD

In the intrablock analysis of variance of an incomplete block design or BIBD, the treatment effects were estimated after eliminating the block effects from the normal equations.

In a way, the block effects were assumed to be not marked enough and so they were eliminated.

It is possible in many situations that the block effects are influential and marked.

In such situations, the block totals may carry information about the treatment combinations also.

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This information can be used in estimating the treatment effects which may provide more efficient results.

This is accomplished by an interblock analysis of BIBD and used further through the recovery of interblock information.

So we first conduct the interblock analysis of BIBD.

We do not derive the expressions a fresh but we use the assumptions and results from the interblock analysis of an incomplete block design.

We additionally assume that the block effects are random with variance σ_β^2.

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After estimating the treatment effects under interblock analysis, we use the results for the pooled estimation and recovery of interblock information in a BIBD.

In case of BIBD,

$$N'N = \begin{pmatrix} \sum_i n_{i1}^2 & \sum_i n_{i1}n_{i2} & \cdots \sum_i n_{i1}n_{iv} \\ \sum_i n_{i1}n_{i2} & \sum_i n_{i2}^2 & \cdots & \sum_i n_{i2}n_{iv} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_i n_{iv}n_{i1} & \sum_i n_{iv}n_{i2} & \cdots & \sum_i n_{iv}^2 \end{pmatrix} = \begin{pmatrix} r & \lambda \cdots \lambda \\ \lambda & r \cdots \lambda \\ \vdots & \vdots \ddots \vdots \\ \lambda & \lambda \cdots r \end{pmatrix} = (r-\lambda)I_v + \lambda E_{v1}E_{v1}'$$

$$(N'N)^{-1} = \frac{1}{r-\lambda}\left[I_v - \frac{\lambda E_{v1}E_{v1}'}{rk} \right]$$

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The interblock estimate of τ can be obtained by substituting the expression on $(N'N)^{-1}$ in the earlier obtained interblock estimate.

$$\tilde{\tau} = (N'N)^{-1} N'B - \frac{GE_{v1}}{bk}.$$

Our next objective is to use the intrablock and interblock estimates of treatment effects together to find an improved estimate of treatment effects.

In order to use the interblock and intrablock estimates of τ together through pooled estimate, we consider the interblock and intrablock estimates of the treatment contrast.

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The intrablock estimate of treatment contrast $l'\tau$ **is**

$$l'\hat{\tau} = l'C^{-}Q = \frac{k}{\lambda v} l'Q = \frac{k}{\lambda v}\sum_j l_j Q_j = \sum_j l_j \hat{\tau}_j, \text{ say.}$$

The interblock estimate of treatment contrast $l'\tau$ **is**

$$l'\tilde{\tau} = \frac{l'N'B}{r-\lambda} \quad (\text{since } l'E_{v1} = 0)$$

$$= \frac{1}{r-\lambda}\sum_{j=1}^{v} l_j \left(\sum_{i=1}^{b} n_{ij} B_i\right) = \frac{1}{r-\lambda}\sum_{j=1}^{v} l_j T_j = \sum_{j=1}^{v} l_j \tilde{\tau}_j.$$

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The variance of $l'\hat{\tau}$ is obtained as

$$Var(l'\hat{\tau}) = \left(\frac{k}{\lambda v}\right)^2 Var\left(\sum_j l_j Q_j\right)$$

$$= \left(\frac{k}{\lambda v}\right)^2 \left[\sum_j l_j^2 Var(Q_j) + 2\sum_j \sum_{j'(\neq j)} l_j l_{j'} Cov(Q_j, Q_{j'})\right].$$

Since

$$Var(Q_j) = r\left(1 - \frac{1}{k}\right)\sigma^2,$$

$$Cov(Q_j, Q_{j'}) = -\frac{\lambda}{k}\sigma^2, \quad (j \neq j'),$$

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so

$$Var(l'\hat{\tau}) = \left(\frac{k}{\lambda v}\right)^2 \left[r\left(1-\frac{1}{k}\right)\sigma^2 \sum_j l_j^2 - \frac{\lambda}{k}\left\{ \left(\sum_j l_j\right)^2 - \sum_j l_j^2 \right\}\sigma^2 \right]$$

$$= \left(\frac{k}{\lambda v}\right)^2 \left[\frac{r(k-1)}{k} \sum_j l_j^2 + \frac{\lambda}{k} \sum_j l_j^2 \right] \sigma^2 \quad \textbf{(since } \sum_{\mathbf{j}} \ell_{\mathbf{j}} = 0 \textbf{ being contrast)}$$

$$= \left(\frac{k}{\lambda v}\right)^2 \frac{1}{k} \left[\lambda(v-1) + \lambda\right] \sum_j l_j^2 \, \sigma^2 \qquad \textbf{(using } r(k-1) = \lambda(v-1)\textbf{)}$$

$$= \left(\frac{k}{\lambda v}\right) \sigma^2 \sum_j l_j^2 .$$

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Similarly, the variance of $\ell'\tilde{\tau}$ is obtained as

$$Var(l'\tilde{\tau}) = \left(\frac{1}{r-\lambda}\right)^2 \left[\sum_j l_j^2 Var(T_j) + 2\sum_j \sum_{j'(\neq j)} l_j l_{j'} Cov(T_j, T_{j'})\right]$$

$$= \left(\frac{1}{r-\lambda}\right)^2 \left[r\sigma_f^2 \sum_j l_j^2 + \lambda\sigma_f^2 \left\{\left(\sum_j l_j\right)^2 - \sum_j l_j^2\right\}\right]$$

$$= \frac{\sigma_f^2}{r-\lambda}\sum_j l_j^2.$$

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The information on $\ell'\hat{\tau}$ and $\ell'\tilde{\tau}$ can be used together to obtain a more efficient estimator of $\ell'\tau$ by considering the weighted arithmetic mean of $\ell'\hat{\tau}$ and $\ell'\tilde{\tau}$.

This will be the minimum variance unbiased and estimator of $\ell'\tau$ when the weights of the corresponding estimates are chosen such that they are inversely proportional to the respective variances of the estimators.

Thus the weights to be assigned to intrablock and interblock estimates are reciprocal to their variances as

$\lambda v/(k\sigma^2)$ and $(r-\lambda)/\sigma_f^2$, respectively.

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Then the pooled mean of these two estimators is

$$L^* = \frac{\dfrac{\lambda v}{k\sigma^2}\sum_j l_j \hat{\tau}_j + \dfrac{r-\lambda}{\sigma_f^2}\sum_j l_j \tilde{\tau}_j}{\dfrac{\lambda v}{k\sigma^2} + \dfrac{r-\lambda}{\sigma_f^2}}$$

$$= \frac{\dfrac{\lambda v \omega_1}{k}\sum_j l_j \hat{\tau}_j + (r-\lambda)\omega_2 \sum_j l_j \tilde{\tau}_j}{\dfrac{\lambda v}{k}\omega_1 + (r-\lambda)\omega_2} = \frac{\lambda v \omega_1 \sum_j l_j \hat{\tau}_j + k(r-\lambda)\omega_2 \sum_j l_j \tilde{\tau}_j}{\lambda v \omega_1 + k(r-\lambda)\omega_2}$$

$$= \sum_j l_j \left[\frac{\lambda v \omega_1 \hat{\tau}_j + k(r-\lambda)\omega_2 \tilde{\tau}_j}{\lambda v \omega_1 + k(r-\lambda)\omega_2} \right] = \sum_j l_j \tau_j^*$$

where $\tau_j^* = \dfrac{\lambda v \omega_1 \hat{\tau}_j + k(r-\lambda)\omega_2 \hat{\tau}_j}{\lambda v \omega_1 + k(r-\lambda)\omega_2}$**,** $\quad \omega_1 = \dfrac{1}{\sigma^2}, \quad \omega_2 = \dfrac{1}{\sigma_f^2}.$

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Now we simplify the expression of τ_j^* so that it becomes more compatible in further analysis.

Since $\hat{\tau}_j = (k/\lambda v)Q_j$ **and** $\tilde{\tau}_j = T_j/(r-\lambda)$, **so the numerator of** τ_j^* **can be expressed as**

$$\omega_1 \lambda v \hat{\tau}_j + \omega_2 k(r-\lambda)\tilde{\tau}_j = \omega_1 k Q_j + \omega_2 k T_j$$

Similarly, the denominator of τ_j^* **can be expressed as**

$$\begin{aligned}
&\omega_1 \lambda v + \omega_2 k(r-\lambda) \\
&= \omega_1 \left[\frac{vr(k-1)}{v-1}\right] + \omega_2 \left[k\left(r - \frac{r(k-1)}{v-1}\right)\right] \quad \text{(using } \lambda(v-1) = r(k-1)) \\
&= \frac{1}{v-1}\left[\omega_1 vr(k-1) + \omega_2 kr(v-k)\right].
\end{aligned}$$

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Let $W_j^* = (v-k)V_j - (v-1)T_j + (k-1)G$

where $\sum_j W_j^* = 0.$ **Using these results we have**

$$\tau_j^* = \frac{(v-1)\left[\omega_1 k Q_j + \omega_2 k T_j\right]}{\omega_1 r v(k-1) + \omega_2 k r(v-k)}$$

$$= \frac{(v-1)\left[\omega_1 (kV_j - T_j) + \omega_2 k T_j\right]}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]} \quad \textbf{(using } Q_j = V_j - \frac{T_j}{k}\textbf{)}$$

$$= \frac{\omega_1 k(v-1)V_j + (k\omega_2 - \omega_1)(v-1)T_j}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]}$$

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$$= \frac{\omega_1 k(v-1)V_j + (\omega_1 - k\omega_2)\left[W_j^* - (v-k)V_j - (k-1)G\right]}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]}$$

$$= \frac{\left[\omega_1 k(v-1) - (\omega_1 - k\omega_2)(v-k)\right]V_j + (\omega_1 - k\omega_2)\left[W_j^* - (k-1)G\right]}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]}$$

$$= \frac{1}{r}\left[V_j + \frac{\omega_1 - k\omega_2}{\omega_1 v(k-1) + \omega_2 k(v-k)}\left\{W_j^* - (k-1)G\right\}\right]$$

$$= \frac{1}{r}\left[V_j + \xi\left\{W_j^* - (k-1)G\right\}\right]$$

where $\xi = \dfrac{\omega_1 - k\omega_2}{\omega_1 v(k-1) + \omega_2 k(v-k)}, \quad \omega_1 = \dfrac{1}{\sigma^2}, \quad \omega_2 = \dfrac{1}{\sigma_f^2}.$

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Thus the pooled estimate of the contrast $l'\tau$ is

$$l'\tau^* = \sum_j l_j \tau_j^* = \frac{1}{r}\sum_j l_j (V_j + \xi W_j^*) \quad \textbf{(since } \sum_j l_j = 0 \textbf{ being contrast)}$$

The variance of $l'\tau^*$ is

$$Var(l'\tau^*) = \frac{k}{\lambda v \omega_1 + k(r-\lambda)\omega_2}\sum_j l_j^2$$

$$= \frac{k(v-1)}{r\left[v(k-1)\omega_1 + k(v-k)\omega_2\right]}\sum_j l_j^2 \qquad \textbf{(using } \lambda(v-1) = r(k-1)\textbf{)}$$

$$= \sigma_E^2 \frac{\sum_j l_j^2}{r}$$

where $\sigma_E^2 = \dfrac{k(v-1)}{v(k-1)\omega_1 + k(v-k)\omega_2}$ **is called as the effective variance.**

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Note that the variance of any elementary contrast based on the pooled estimates of the treatment effects is

$$Var(\tau_j^* - \tau_{j'}^*) = \frac{2}{r}\sigma_E^2.$$

The effective variance can be approximately estimated by

$$\hat{\sigma}_E^2 = MSE\left[1 + (v-k)\omega^*\right]$$

where MSE is the mean square due to error obtained from the intrablock analysis as

$$MSE = \frac{SS_{Error(t)}}{bk - b - v + 1}$$

and

$$\omega^* = \frac{\omega_1 - \omega_2}{v(k-1)\omega_1 + k(v-k)\omega_2}.$$

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The quantity ω^* depends upon the unknown σ^2 and σ_β^2.

To obtain an estimate of ω^*, we can obtain the unbiased estimates of σ^2 and σ_β^2, and then substitute them back in place of σ^2 and σ_β^2 in ω^*. To do this, we proceed as follows.

An estimate of ω_1 can be obtained by estimating σ^2 from the intrablock analysis of variance as $\hat{\omega}_1 = \dfrac{1}{\hat{\sigma}^2} = [MSE]^{-1}$

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The estimate of ω_2 depends on $\hat{\sigma}^2$ and $\hat{\sigma}_\beta^2$. To obtain an unbiased estimator of σ_β^2, consider

$$SS_{Block(adj)} = SS_{Treat(adj)} + SS_{Block(unadj)} - SS_{Treat(unadj)}$$

for which

$$E(SS_{Block(adj)}) = (bk - v)\sigma_\beta^2 + (b-1)\sigma^2.$$

Thus an unbiased estimator of σ_β^2 is

$$\hat{\sigma}_\beta^2 = \frac{1}{bk-v}\left[SS_{Block(adj)} - (b-1)\hat{\sigma}^2\right] = \frac{1}{bk-v}\left[SS_{Block(adj)} - (b-1)MSE\right]$$

$$= \frac{b-1}{bk-v}\left[MS_{Block(adj)} - MSE\right] = \frac{b-1}{v(r-1)}\left[MS_{Block(adj)} - MSE\right]$$

where

$$MS_{Block(adj)} = \frac{SS_{Block(adj)}}{b-1}.$$

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Thus

$$\hat{\omega}_2 = \frac{1}{k\hat{\sigma}^2 + \hat{\sigma}_\beta^2}$$

$$= \frac{1}{v(r-1)\left[k(b-1)SS_{Block(adj)} - (v-k)SS_{Error(t)}\right]}.$$

Recall that our main objective is to develop a test of hypothesis for $H_0 : \tau_1 = \tau_2 = ... = \tau_v$ **and we now want to develop it using the information based on both interblock and intrablock analysis.**

To test the hypothesis related to treatment effects based on the pooled estimate, we proceed as follows.

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Consider the adjusted treatment totals based on the intrablock

and the interblock estimates as

$$T_j^* = T_j + \omega^* W_j^*; \; j = 1, 2, ..., v$$

and use it as usual treatment total as in earlier cases.

The sum of squares due to T_j^* is

$$S_{T^*}^2 = \sum_{j=1}^{v} T_j^{*2} - \frac{\left(\sum_{j=1}^{v} T_j^* \right)^2}{v}.$$

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Note that in the usual analysis of variance technique, the test statistic for such hull hypothesis is developed by taking the ratio of the sum of squares due to treatment divided by its degrees of freedom and the sum of squares due to error divided by its degrees of freedom.

Following the same idea, we define the statistics

$$F^* = \frac{S_{T*}^2 / [(v-1)r]}{MSE[1+(v-k)\hat{\omega}^*]}$$

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$$F^* = \frac{S_{T^*}^2 / [(v-1)r]}{MSE[1+(v-k)\hat{\omega}^*]}$$

where $\hat{\omega}^*$ is an estimator of ω^*. It may be noted that *F depends on $\hat{\omega}^*$.**

The value of $\hat{\omega}^*$ itself depends on the estimated variances $\hat{\sigma}^2$ and $\hat{\sigma}_f^2$.

So it cannot be ascertained that the statistic *F necessary follow the *F* distribution.**

Since the construction of *F is based on the earlier approaches where the statistic was found to follow the exact *F*-distribution, so based on this idea, the distribution of *F** can be considered to be approximately *F* distributed.**

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Thus the approximate distribution of *F is considered as *F* distribution with (*v* - 1) and $(bk-b-v+1)$ degrees of freedom. Also, $\hat{\omega}^*$ is an estimator of ω^* which is obtained by substituting the unbiased estimators of ω_1 and ω_2 .**

An approximate best pooled estimator of $\sum_{j=1} l_j \tau_j$ is

$$\sum_{j=1}^{v} l_j \frac{V_j + \hat{\xi} W_j}{r}$$

and its variance is approximately estimated by $$\frac{k \sum_j l_j^2}{\lambda v \hat{\omega}_1 + (r-\lambda) k \hat{\omega}_2}.$$

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In case of the resolvable BIBD, $\hat{\sigma}_\beta^2$ can be obtained by using the adjusted block with replications sum of squares from the intrablock analysis of variance.

If sum of squares due to such block total is SS^*_{Block} and corresponding mean square is

$$MS^*_{Block} = \frac{SS^*_{Block}}{b-r}$$

Then

$$E(MS^*_{Block}) = \sigma^2 + \frac{(v-k)(r-1)}{b-r}\sigma_\beta^2$$

$$= \sigma^2 + \frac{(r-1)k}{r}\sigma_\beta^2$$

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Thus $k(b-r)=r(v-k)$

$$E\left[rMS^{*}_{Block}-MSE\right]=(r-1)(\sigma^2+k\sigma^2_{\beta})$$

and hence

$$\hat{\omega}_2=\left[\frac{rMS^{*}_{block}-MSE}{r-1}\right]^{-1},$$

$$\hat{\omega}_1=\left[MSE\right]^{-1}.$$

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The increase in the precision using interblock analysis as compared to intrablock analysis is

$$\frac{Var(\hat{\tau})}{Var(\tau^*)} - 1 = \frac{\lambda v \omega_1 + \omega_2 k(r-\lambda)}{\lambda v \omega_1} - 1 = \frac{\omega_2 (r-\lambda)k}{\lambda v \omega_1}.$$

Such an increase may be estimated by $\dfrac{\hat{\omega}_2 (r-\lambda)k}{\lambda v \hat{\omega}_1}$.

Although $\omega_1 > \omega_2$ **but this may not hold true for** $\hat{\omega}_1$ **and** $\hat{\omega}_2$ **.**

The estimates $\hat{\omega}_1$ **and** $\hat{\omega}_2$ **may be negative also and in that case we take** $\hat{\omega}_1 = \hat{\omega}_2$.