Analysis of Variance and Design of Experiments

2^n Factorial Experiments

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Lecture 35

ANOVA in 2^2 Factorial Experiment

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Factorial experiments:

If the number of levels for each factor is the same, we call it is a symmetrical factorial experiment.

If the number of levels of each factor is not the same, then we call it as asymmetrical or mixed factorial experiment.

We consider only symmetrical factorial experiments.

Factorial experiments:

Through the factorial experiments, we can study

- **the individual effect of each factor and**
- **interaction effect.**

An important point to remember is that the factorial experiments are conducted in the design of an experiment. For example, the factorial experiment is conducted as an RBD.

Factorial experiments:

General notation for representing the factors is to use capital letters, e.g., *A,B,C* etc. and levels of a factor are represented in small letters.

For example, if there are two levels of *A,* they are denoted as a_0 and a_1 .

Similarly, the two levels of *B* are represented as b_0 and b_1 .

Another alternative representation to indicate the two levels of *A* is 0 (for a_0) and 1 (for a_1).

The factors of *B* are then 0 (for b_0) and 1 (for b_1).

2^2 Factorial Experiment:

Treating (*ab*) as (*a*) (*b*) symbolically (mathematically and conceptually, it is incorrect), we can now express all the main effects, interaction effect and general mean effect as follows:

2^2 Factorial Experiment:

Main effect of $A = \frac{(a)+(ab)}{2} - \frac{(1)+(b)}{2} = \frac{1}{2}\left[(ab)-(b)+(a)-(1)\right] = \frac{(a-1)(b+1)}{2}$

Main effect of $B = \frac{(b)+(ab)}{2} - \frac{(1)+(a)}{2}$

$$= \frac{1}{2}\left[(ab)-(a)+(b)-(1)\right] = \frac{(a+1)(b-1)}{2}$$

Interaction effect of A **and** $B = \frac{(ab)-(b)}{2} - \frac{(a)-(1)}{2}$

$$= \frac{1}{2}\left[(ab)-(a)+(1)-(b)\right] = \frac{(a-1)(b-1)}{2}$$

General mean effect $(M) = \frac{(1)+(a)+(b)+(ab)}{4}$

$$= \frac{1}{4}\left[(1)+(a)+(b)+(ab)\right] = \frac{(a+1)(b+1)}{4}$$

2^2 Factorial Experiment:

Notice the roles of + and – signs as well as the divisor.

- **There are two effects related to *A* and *B*.**

- **To obtain the effect of a factor, write the corresponding factor with – sign and others with + sign. For example, in the main effect of *A*, *a* occurs with – sign as in $(a - 1)$ and *b* occurs with + sign as in $(b + 1)$.**

- **In *AB*, both the effects are present so *a* and *b* both occur with + signs as in $(a + 1)(b + 1)$.**

2^2 Factorial Experiment:

- **Also note that the main and interaction effects are obtained by considering the typical differences of averages, so they have divisor 2 whereas the general mean effect is based on all the treatment combinations and so it has divisor 4.**

- **There is a well defined statistical theory behind this logic but this logic helps in writing the final treatment combination easily. This is demonstrated later with appropriate reasoning.**

2^2 Factorial Experiment:

These effects can be represented in the following table

Factorial effects	Treatment combinations				Divisor
	(1)	(*a*)	(*b*)	(*ab*)	
M	+	+	+	+	4
A	-	+	-	+	2
B	-	-	+	+	2
AB	+	-	-	+	2

The model corresponding to 2^2 factorial experiment is

$$y_{ijk} = \mu + A_i + B_j + (AB)_{ij} + \varepsilon_{ijk},\ i = 1, 2,\ j = 1, 2,\ k = 1, 2, \ldots, n$$

where *n* observations are obtained for each treatment combinations.

Factorial Experiment:

Example:

Two factors: Irrigation (*I*) and Nitrogen (*N*).

Levels:

Irrigation has 2 levels: I_0 and I_1

Nitrogen has 3 levels: N_0, N_1 and N_2

Factorial Experiment:

When the experiments are conducted factor by factor, then much more resources are required in comparison to the factorial experiment. For example, if we conduct *RBD* for three-levels of Nitrogen N_0, N_1 and N_2 and two levels of irrigation I_0 and I_1, then to have 10 degrees of freedom for the error variance, we need

- **6 replications on nitrogen**
- **11 replications on irrigation.**

So the total number of plots needed is 40.

2^2 Factorial Experiment:

For the factorial experiment with 6 combinations of 2 factors, the total number of plots needed are 18 for the same precision.

We have considered the situation up to now by assuming only one observation for each treatment combination, i.e., no replication.

If *r* replicated observations for each of the treatment combinations are obtained, then the expressions for the main and interaction effects can be expressed as

2^2 Factorial Experiment:

$$A = \frac{1}{2r}\left[(ab) + (a) - (b) - (1)\right]$$

$$B = \frac{1}{2r}\left[(ab) + (b) - (a) - (1)\right]$$

$$AB = \frac{1}{2r}\left[(ab) + (1) - (a) - (b)\right]$$

$$M = \frac{1}{4r}\left[(ab) + (a) + (b) + (1)\right].$$

Now we detail the statistical theory and concepts related to these expressions.

2^2 Factorial Experiment:

Let $Y_* = ((1),\ a,\ b,\ ab)'$ **be the vector of total response values. Then**

$$A = \frac{1}{2r}\ell'_A Y_* = \frac{1}{2r}(-1 \quad 1 \quad -1 \quad 1)Y_*$$

$$B = \frac{1}{2r}\ell'_B Y_* = \frac{1}{2r}(-1 \quad -1 \quad 1 \quad 1)Y_*$$

$$AB = \frac{1}{2r}\ell'_{AB} Y_* = \frac{1}{2r}(1 \quad -1 \quad -1 \quad 1)Y_*.$$

Note that *A, B* and *AB* are the linear contrasts.

Recall that a linear parametric function is estimable only when it is in the form of linear contrast.

Moreover, *A, B* and *AB* are the linear orthogonal contrasts in the total response values (1), *a*, *b*, *ab* except for the factor 1/2*r*.

2^2 Factorial Experiment:

The sum of squares of a linear parametric function *l'y* is given by

$$\frac{(\ell' y)^2}{\ell' \ell}.$$

If there are *r* replicates, then the sum of squares is $\frac{(\ell' y)^2}{r\ell' \ell}$ **.**

It may also be recalled under the normality of *y*'s, this sum of squares has a Chi-square distribution with one degree of freedom (χ_1^2)**.**

2^2 Factorial Experiment:

Thus the various associated sum of squares due to *A*, *B* and *AB* are given by the following:

$$SSA = \frac{(\ell'_A Y_*)^2}{r\ell'_A \ell_A} = \frac{1}{4r}(ab + a - b - (1))^2$$

$$SSB = \frac{(\ell'_B Y_*)^2}{r\ell'_B \ell_B} = \frac{1}{4r}(ab + b - a - (1))^2$$

$$SSAB = \frac{(\ell'_{AB} Y_*)^2}{r\ell'_{AB} \ell_{AB}} = \frac{1}{4r}(ab + (1) - a - b)^2.$$

Each of *SSA, SSB* and *SSAB* has χ_1^2 under normality of Y_*.

The sum of squares due to total is computed as usual

$$TSS = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{r} y_{ijk}^2 - \frac{G^2}{4r}$$

where $G = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{r} y_{ijk}$ is the grand total of all the observations.

2^2 Factorial Experiment:

The *TSS* has χ^2 **distribution with** $(2^2 r - 1)$ **degrees of freedom.**

The sum of squares due to error is also computed as usual as

$$SSE = TSS - SSA - SSB - SSAB$$

which has χ^2 **distribution with** $(4r-1)-1-1-1 = 4(r-1)$ **degrees of freedom.**

The mean squares are

$$MSA = \frac{SSA}{1}, \qquad MSB = \frac{SSB}{1},$$

$$MSAB = \frac{SSAB}{1}, \qquad MSE = \frac{SSA}{4(r-1)}.$$

2^2 Factorial Experiment:

The *F* - statistic corresponding to *A, B* and *AB* are

$$F_A = \frac{MSA}{MSE} \sim F(1, 4(r-1) \text{ under } H_0,$$

$$F_B = \frac{MSB}{MSE} \sim F(1, 4(r-1) \text{ under } H_0,$$

$$F_{AB} = \frac{MSAB}{MSE} \sim F(1, 4(r-1) \text{ under } H_0.$$

The decision rule is to reject the concerned null hypothesis when the value of the concerned *F* statistic

$$F_{\text{effect}} > F_{1-\alpha}(1, 4(r-1)).$$

2^2 Factorial Experiment:

The ANOVA table is case of 2^2 factorial experiment is given as follows:

Source	Sum of squares	Degrees of freedom	Mean squares	F
A	SSA	1	MSA	$F_A = \frac{MSA}{MSE}$
B	SSB	1	MSB	$F_B = \frac{MSB}{MSE}$
AB	$SSAB$	1	$MSAB$	$F_{AB} = \frac{MSAB}{MSE}$
$Error$	SSE	$4(r-1)$	MSE	
Total	TSS	$4r-1$		