

Analysis of Variance and Design of Experiments

2^n Factorial Experiments

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Lecture 36

ANOVA in 2^3 and 2^n Factorial Experiment



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Slides can be downloaded from <http://home.iitk.ac.in/~shalab/sp1>

2³ Factorial Experiment:

Suppose that in a complete factorial experiment, there are three factors - A , B and C , each at two levels, viz., a_0, a_1 ; b_0, b_1 and c_0, c_1 respectively. There are a total of eight number of combinations:

$$a_0b_0c_0, a_0b_0c_1, a_0b_1c_0, a_0b_1c_1, \\ a_1b_0c_0, a_1b_0c_1, a_1b_1c_0, a_1b_1c_1.$$

Each treatment combination has r replicates, so the total number of observations are $N = 2^3 r = 8r$ that are to be analyzed for their influence on the response.

2³ Factorial Experiment:

Assume the total response values are

$$Y_* = [(1), a, b, ab, c, ac, bc, abc]'$$

The response values can be arranged in a three-dimensional contingency table. The effects are determined by the linear contrasts

$$\ell'_{effect} Y_* = \ell'_{effect} ((1), a, b, ab, c, ac, bc, abc)$$

using the following table:

2³ Factorial Experiment:

Factorial effect	Treatment combinations							
	(1)	<i>a</i>	<i>b</i>	<i>ab</i>	<i>c</i>	<i>ac</i>	<i>bc</i>	<i>abc</i>
<i>I</i>	+	+	+	+	+	+	+	+
<i>A</i>	-	+	-	+	-	+	-	+
<i>B</i>	-	-	+	+	-	-	+	+
<i>AB</i>	+	-	-	+	+	-	-	+
<i>C</i>	-	-	-	-	+	+	+	+
<i>AC</i>	+	-	+	-	-	+	-	+
<i>BC</i>	+	+	-	-	-	-	+	+
<i>ABC</i>	-	+	+	-	+	-	-	+

2³ Factorial Experiment:

Note that once a few rows have been determined in this table, rest can be obtained by simple multiplication of the symbols.

For example, consider the column corresponding to a , we note that

A has + sign, B has – sign ,

so AB has – sign (sign of A x sign of B).

Once AB has - sign,

C has – sign, then ABC has (sign of AB X sign of C) which is + sign and so on.

2³ Factorial Experiment:

The first row is a basic element. With this $a = 1'Y_*$ can be computed where **1** is a column vector of all elements unity.

If other rows are multiplied with the first row, they stay unchanged (therefore we call it as identity and denoted as *I*).

Every other row has the same number of + and – signs. If + is replaced by 1 and – is replaced by -1, we obtain the vectors of orthogonal contrasts with the norm 8 (= 2³).

If each row is multiplied by itself, we obtain *I* (first row). The product of any two rows leads to a different row in the table.

2³ Factorial Experiment:

For example

$$A.B = AB$$

$$AB.B = AB^2 = A$$

$$AC.BC = A.C^2B = AB.$$

The structure in the table helps in estimating the average effect.

For example, the average effect of *A* is

$$A = \frac{1}{4r} [(a) - (1) + (ab) - (b) + (ac) - (c) + (abc) - (bc)]$$

which has the following explanation.

2³ Factorial Experiment:

(i) Average effect of A at low level of B and C

$$\equiv (a_1b_0c_0) - (a_0b_0c_0) \equiv \frac{[(a) - (1)]}{r}$$

(ii) Average effect of A at high level of B and low level of C

$$\equiv (a_1b_1c_0) - (a_0b_1c_0) \equiv \frac{[(ab) - (b)]}{r}$$

(iii) Average effect of A at low level of B and high level of C

$$\equiv (a_1b_0c_1) - (a_0b_0c_1) \equiv \frac{[(ac) - (c)]}{r}$$

(iv) Average effect of A at high level of B and C

$$\equiv (a_1b_1c_1) - (a_0b_1c_1) \equiv \frac{[(abc) - (bc)]}{r}.$$

Hence for all combinations of B and C , the average effect of A is the average of all the average effects in (i)-(iv).

2³ Factorial Experiment:

Similarly, other main and interaction effects are as follows:

$$B = \frac{1}{4r} [(b) + (ab) + (bc) + (abc) - (1) - (a) - (c) - (ac)] = \frac{(a+1)(b-1)(c+1)}{4r}$$

$$C = \frac{1}{4r} [c + (ac) + (bc) + (abc) - (1) - (a) - (b) - (ab)] = \frac{(a+1)(b+1)(c-1)}{4r}$$

$$AB = \frac{1}{4r} [(1) + (ab) + (c) + (abc) - (a) - (b) - (ac) - (bc)] = \frac{(a-1)(b-1)(c+1)}{4r}$$

$$AC = \frac{1}{4r} [(1) + (b) + (ac) + (abc) - (a) - (ab) - (c) - (bc)] = \frac{(a-1)(b+1)(c-1)}{4r}$$

$$BC = \frac{1}{4r} [(1) + (a) + (bc) + (abc) - (b) - (ab) - (c) - (ac)] = \frac{(a+1)(b-1)(c-1)}{4r}$$

$$ABC = \frac{1}{4r} [(abc) + (ab) + (b) + (c) - (ab) - (ac) - (bc) - (1)] = \frac{(a-1)(b-1)(c-1)}{4r}.$$

2³ Factorial Experiment:

Various sum of squares in the 2³ factorial experiment are obtained as

$$SS(\text{Effect}) = \frac{(\text{linear contrast})^2}{8r} = \frac{\left(\ell'_{\text{effect}} Y_*\right)^2}{r \ell'_{\text{effect}} \ell_{\text{effect}}}$$

which follow a Chi-square distribution with one degree of freedom under normality of Y_* . The corresponding mean sum of squares is obtained as

$$MS(\text{Effect}) = \frac{SS(\text{Effect})}{\text{Degrees of freedom}}$$

2³ Factorial Experiment:

The corresponding F - statistics are obtained by

$$F_{\text{Effect}} = \frac{MS(\text{Effect})}{MS(\text{Error})}$$

which follows an F - distribution with degrees of freedoms 1 and error degrees of freedom under the respective null hypothesis. The decision rule is to reject the corresponding null hypothesis at the α level of significance whenever

$$F_{\text{effect}} > F_{1-\alpha}(1, df_{\text{error}}).$$

2³ Factorial Experiment:

These outcomes are presented in the following ANOVA table

Sources	Sum of squares	Degrees of freedom	Mean sum of squares	<i>F</i>
<i>A</i>	<i>SSA</i>	1	$MSA = SSA / 1$	F_A
<i>B</i>	<i>SSB</i>	1	$MSB = SSB / 1$	F_B
<i>AB</i>	<i>SSAB</i>	1	$MSAB = SSAB / 1$	F_{AB}
<i>C</i>	<i>SSC</i>	1	$MSC = SSC / 1$	F_C
<i>AC</i>	<i>SSAC</i>	1	$MSAC = SSAC / 1$	F_{AC}
<i>BC</i>	<i>SSBC</i>	1	$MSBC = SSBC / 1$	F_{BC}
<i>ABC</i>	<i>SSABC</i>	1	$MSABC = SSABC / 1$	F_{ABC}
Error	<i>SS(Error)</i>	$8(r - 1)$	$MS(Error) = SS(Error) / \{8(r - 1)\}$	F_{ABC}
Total	TSS	$8r - 1$		

2ⁿ Factorial Experiment:

Based on the theory developed for 2² and 2³ factorial experiments, we now extend them for the 2ⁿ factorial experiment.

- Capital letters *A, B, C, ...* denote the factors. They are the main effect contrast for the factors *A, B, C, ...*
- *AB, AC, BC, ...* denote the first order or 2-factor interactions
- *ABC, ABD, BCD, ...* denote the second-order or 3-factor interactions and so on.
- Each of the main effect and interaction effect carries one degree of freedom.

2ⁿ Factorial Experiment:

- Total number of main effects = $\binom{n}{1} = n.$
- Total number of first-order interactions = $\binom{n}{2}$
- Total number of second-order interactions = $\binom{n}{3}$

and so on.

Standard order for treatment combinations:

The list of treatments can be expressed in a standard order.

- For one factor A , the standard order is $(1), a$.
- For two factors A and B , the standard order is obtained by adding b and ab in the standard order of one factor A . This is derived by multiplying (1) and a by b , i.e.

$$b \times \{(1), a\} = (1), a, b, ab$$

- For three factors, add c, ac, bc and abc which are derived by multiplying the standard order of A and B by c , i.e.

$$c \times \{(1), a, b, ab\} = (1), a, b, ab, c, ac, bc, abc$$

Standard order for treatment combinations:

Thus the standard order of any factor is obtained step by step by multiplying it with an additional letter to preceding standard order.

For example, the standard order of *A,B,C* and *D* is 2^4 factorial experiment is

$$\begin{aligned} & (1), a, b, ab, c, ac, bc, abc, \\ & d \times \{(1), a, b, ab, c, ac, bc, abc\} \\ & = (1), a, b, ab, c, ac, bc, abc, d, \\ & ad, bd, abd, cd, acd, bcd, abcd. \end{aligned}$$

How to find the contrasts for main effects and interaction effect:

Recall that earlier, we had illustrated the concept in writing the contrasts for main and interaction effects. For example, in a 2^2 factorial experiment, we had expressed

$$A = \frac{1}{2}(a-1)(b+1) = \frac{1}{2}[(-1) + (a) - (b) + (ab)]$$

$$AB = \frac{1}{2}(a-1)(b-1) = \frac{1}{2}[(1) - (a) - (b) + (ab)].$$

Note that each effect has two component - divisor and contrast.

How to find the contrasts for main effects and interaction effect:

When the order of factorial increases, it is cumbersome to derive such expressions. Some methods have been suggested to write the expressions for factorial effects. First, we detail how to write divisor and then illustrate the methods for obtaining the contrasts.

How to write divisor:

In a 2^n factorial experiment,

- the general mean effect has divisor 2^n and
- any effect (main or interaction) has divisor 2^{n-1} .

For example, in a 2^6 factorial experiment, the general mean effect has divisor 2^6 and any main effect or interaction effect of any order has divisor $2^{6-1} = 2^5$.

If r replicates of each effect are available, then

- the general mean effect has divisor $r 2^n$ and
- any main effect or interaction effect of any order has a divisor $r 2^{n-1}$.

How to write contrasts: Method 1

Contrast belonging to the main effects and the interaction effects are written as follows:

$$A = (a - 1)(b + 1)(c + 1) \dots (z + 1)$$

$$B = (a + 1)(b - 1)(c + 1) \dots (z + 1)$$

$$C = (a + 1)(b + 1)(c - 1) \dots (z + 1)$$

⋮

$$AB = (a - 1)(b - 1)(c + 1) \dots (z + 1)$$

$$BC = (a + 1)(b - 1)(c - 1) \dots (z + 1)$$

⋮

⋮

$$ABC = (a - 1)(b - 1)(c - 1) \dots (z + 1).$$

⋮

$$ABC \dots Z = (a - 1)(b - 1)(c - 1) \dots (z - 1)$$

How to write contrasts: Method 1

Look at the pattern of assigning + and – signs on the right-hand side. The letters common on left and right hand sides of the equality (=) sign (irrespective of small or capital letters) contain – sign and rest contain + sign.

The expression on right-hand side when simplified algebraically give the contrasts in terms of treatment combination.

How to write contrasts: Method 1

For example, in a 2^3 factorial

$$\begin{aligned}A &= \frac{1}{2^{3-1}}(a-1)(b+1)(c+1) \\ &= \frac{1}{4}[-(1) + (a) - (b) + (ab) - (c) + (ac) - (bc) + (abc)] \\ M &= \frac{1}{2^3}(a+1)(b+1)(c+1) \\ &= \frac{1}{8}[(1) + (a) + (b) + (ab) + (c) + (ac) + (bc) + (abc)]\end{aligned}$$

How to write contrasts: Method 2

- Form a table such that
 - rows correspond to the main or interaction effect and
 - columns correspond to treatment combinations (or another way round)
- + and – signs in the table indicate the sign of the treatment combinations of main and interaction effects.

How to write contrasts: Method 2

- Signs are determined by the “rule of odds and evens” given as follows:
 - if the interaction has an even number of letters (*AB, ABCD,...*), a treatment combination having an even number of letters common with the interaction enters with a + sign and one with an odd number of letters common enters with a – sign.
 - if the interaction has an odd number of letters (*A, ABC, ...*) the rule is reversed.

How to write contrasts: Method 2

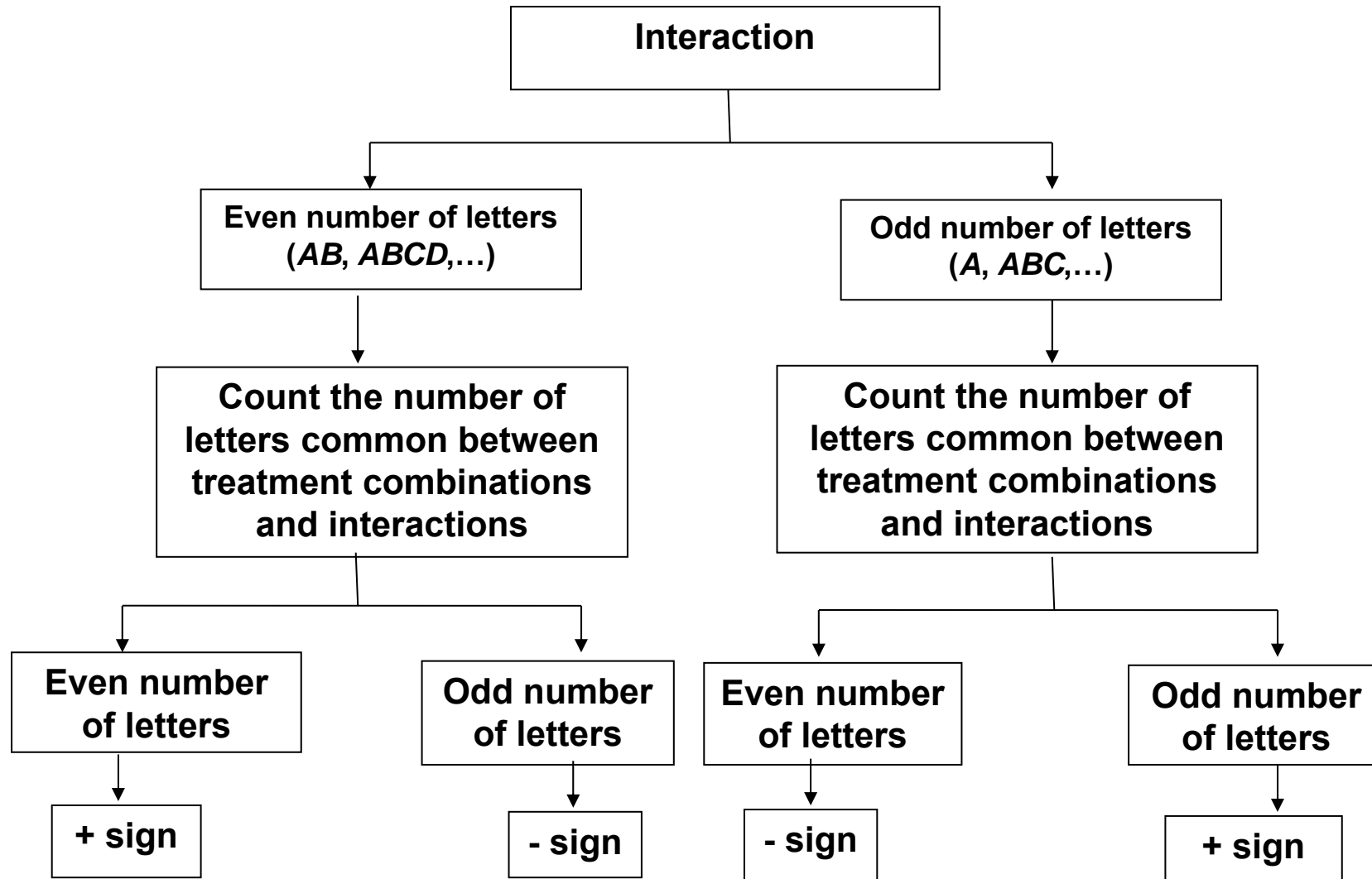
- Once few rows are filled up, others can be obtained through multiplication rule. For example, the sign of $ABCD$ is obtained as

(sign of A x sign of BCD) or (sign of AB x sign of CD).

- Treatment combination (1) is taken to have an even number (zero) of letters common with every interaction.

- This rule of assignment of + or – is illustrated in the following flow diagram:

How to write contrasts: Method 2



How to write contrasts: Method 2

For example, in a 2^3 factorial experiment, write

- rows for main and interaction effects and
- columns for treatment combinations in standard order.
- Take treatment combination (1) to have an even number (zero) of letter common with every interaction.

How to write contrasts: Method 2

This gives the following table

Factorial effect	Treatment combinations							
	(1)	<i>a</i>	<i>b</i>	<i>ab</i>	<i>c</i>	<i>ac</i>	<i>bc</i>	<i>abc</i>
<i>I</i>	+	+	+	+	+	+	+	+
<i>A</i>	-	+	-	+	-	+	-	+
<i>B</i>	-	-	+	+	-	-	+	+
<i>AB</i>	+	-	-	+	+	-	-	+
<i>C</i>	-	-	-	-	+	+	+	+
<i>AC</i>	+	-	+	-	-	+	-	+
<i>BC</i>	+	+	-	-	-	-	+	+
<i>ABC</i>	-	+	+	-	+	-	-	+

Sums of squares:

Suppose 2^n factorial experiment is carried out in a randomized block design with r replicates.

Denote the total yield (output) from r plots (experimental units) receiving a particular treatment combination by the same symbol within a square bracket.

For example, $[ab]$ denotes the total yield from the plots receiving the treatment combination (ab) .

Sums of squares:

In a 2^2 factorial experiment, the factorial effect totals are

$$[A] = [ab] - [b] + [a] - [1]$$

$[ab]$ = treatment total, i.e. the sum of r observations in which both the factors A and B are at the second level.

$[a]$ = treatment total, i.e., the sum of r observations in which factor A is at the second level and factor B is at the first level

$[b]$ = treatment total, i.e., the sum of r observations in which factor A is at the first level and factor B is at the second level.

$[1]$ = treatment total, i.e. the sum of r observations in which both the factors A and B are at the first level.

Sums of squares:

$$[A] = \sum_{i=1}^r [y_{i(ab)} - y_{i(b)} + y_{i(a)} - y_{i(1)}]$$
$$= \ell'_A y_A \text{ (say).}$$

where l_A is a vector of +1 and 1 and y_A is a vector denoting the responses from ab , b , a and 1. Similarly, other effects can also be found.

Similarly, other effects can also be found.

The sum of squares due to a particular effect is obtained as

$$\frac{[\text{Total yield}]^2}{\text{Total number of observations}}$$

Sums of squares:

In a 2^2 factorial experiment in an RBD, the sum of squares due to A is

$$SSA = \frac{(\ell'_A y_A)^2}{r2^2}.$$

In a 2^n factorial experiment in an RBD, the divisor will be $r.2^n$.
If Latin square design is used based on $2^n \times 2^n$ Latin square, then r is replaced by 2^n .