Analysis of Variance and Design of Experiments

Confounding in 2^n Factorial Experiments

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Lecture 37

Understanding Confounding in 2^2 Factorial Experiment

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Need of Confounding:

If the number of factors or levels increase in a factorial experiment, then the number of treatment combinations increases rapidly.

When the number of treatment combinations is large, then it may be difficult to get the blocks of sufficiently large size to accommodate all the treatment combinations.

Need of Confounding:

Under such situations, one may use either connected incomplete block designs, e.g., balanced incomplete block designs (BIBD) where all the main effects and interaction contrasts can be estimated

or

use unconnected designs where not all these contrasts can be estimated.

Non-estimable contrasts are said to be confounded.

Need of Confounding:

Note that a linear function $\lambda'\beta$ is said to be estimable if there exist a linear function *l'y* of the observations on random variable *y* such that $E(l'y) = \lambda'\beta.$

Now there arise two questions.

Firstly, what does confounding means and

secondly, how does it compares to using BIBD.

Need of Confounding: Example

In order to understand the confounding, let us consider a simple example of 2^2 factorial with factors A and B.

The four treatment combinations are (1), a, b and ab.

Suppose each batch of raw material to be used in the experiment is enough only for two treatment combinations to be tested.

So two batches of raw material are required.

Thus two out of four treatment combinations must be assigned to each block.

Confounding example in 2^2 factorial experiment:

Suppose this 2^2 factorial experiment is being conducted in a randomized block design. Then the corresponding model is

$$E(y_{ij}) = \mu + \beta_i + \tau_j, \quad i = 1, 2, \; j = 1, 2, 3, 4$$

with

$$A = \frac{1}{2r}\left[ab + a - b - (1)\right],$$

$$B = \frac{1}{2r}\left[ab + b - a - (1)\right],$$

$$AB = \frac{1}{2r}\left[ab + (1) - a - b\right].$$

Confounding example in 2^2 factorial experiment:

Suppose the following block arrangement is opted:

Block 1	Block 2
(1)	*a*
ab	*b*

The block effects of blocks 1 and 2 are β_1 and β_2, respectively, then the average responses corresponding to treatment combinations *a*, *b*, *ab* and (1) are

$$E\left[y(1)\right]=\mu+\beta_1+\tau(1), \qquad E\left[y(ab)\right]=\mu+\beta_1+\tau(ab),$$

$$E\left[y(a)\right]=\mu+\beta_2+\tau(a), \qquad E\left[y(b)\right]=\mu+\beta_2+\tau(b),$$ **respectively.**

Here *y*(*a*), *y*(*b*), *y*(*ab*), *y*(1) and $\tau(a),\ \tau(b),\ \tau(ab),\ \tau(1)$ denote the responses and treatments corresponding to *a*, *b*, *ab* and (1), respectively.

Confounding example in 2^2 factorial experiment:

Ignoring the factor 1/2*r* in *A*, *B*, *AB* (just for simplicity in understanding) and using $E[y(a)],\ E[y(b)],\ E[y(ab)],\ E(y(1)]$ **, the effect *A* is expressible as follows :**

$$A = [\mu + \beta_1 + \tau(ab)] + [\mu + \beta_2 + \tau(a)] - [\mu + \beta_2 + \tau(b)] - [\mu + \beta_1 + \tau(1)]$$

$$= \tau(ab) + \tau(a) - \tau(b) - \tau(1).$$

So the block effect is not present in *A* and it is not mixed up with the treatment effects.

In this case, we say that the main effect *A* is not confounded.

Confounding example in 2^2 factorial experiment:

Similarly, for the main effect *B,* we have

$$B = [\mu + \beta_1 + \tau(ab)] + [\mu + \beta_2 + \tau(b)] - [\mu + \beta_2 + \tau(a)] - [\mu + \beta_1 + \tau(1)]$$

$$= \tau(ab) + \tau(b) - \tau(a) - \tau(1).$$

So there is no block effect present in *B* and thus *B* is not confounded.

Confounding example in 2^2 factorial experiment:

For the interaction effect *AB*, we have

$$AB = [\mu+\beta_1+\tau(ab)]+[\mu+\beta_1+\tau(1)]-[\mu+\beta_2+\tau(a)]-[\mu+\beta_2+\tau(b)]$$

$$= 2(\beta_1-\beta_2)+\tau(ab)+\tau(1)-\tau(a)-\tau(b).$$

Here the block effects are present in *AB*. In fact, the block effects are β_1 and β_2 are mixed up with the treatment effects and cannot be separated individually from the treatment effects in *AB*.

So *AB* is said to be confounded (or mixed up) with the blocks.

Confounding example in 2^2 factorial experiment:

Alternatively, if the arrangement of treatments in blocks are as follows:

Block 1	**Block 2**
(1)	***a***
b	***ab***

then the main effect A is expressible as

$$A = [\mu + \beta_1 + \tau(ab)] + [\mu + \beta_1 + \tau(a)] - [\mu + \beta_2 + \tau(b)] - [\mu + \beta_2 + \tau(1)]$$

$$= 2(\beta_1 - \beta_2) + \tau(ab) + \tau(a) - \tau(b) - \tau(1)$$

Observe that the block effects β_1 and β_2 are present in this expression. So the main effect A is confounded with the blocks in this arrangement of treatments.

Confounding example in 2^2 factorial experiment:

We notice that it is in our control to decide that which of the effect is to be confounded.

The order in which treatments are run in a block is determined randomly.

The choice of block to be run first is also randomly decided.

Confounding example in 2^2 factorial experiment:

The following observation emerges from the allocation of treatments in blocks:

"For a given effect, when two treatment combinations with the same signs are assigned to one block and the other two treatment combinations with the same but opposite signs are assigned to another block, then the effect gets confounded".

Confounding example in 2^2 factorial experiment:

For example, in case *AB* is confounded, then

- ***ab*** **and (1) with + signs are assigned to block 1 whereas**
- ***a*** **and *b* with – signs are assigned to block 2.**

Block 1	Block 2
(1)	*a*
ab	*b*

$$AB = [\mu + \beta_1 + \tau(ab)] + [\mu + \beta_1 + \tau(1)] - [\mu + \beta_2 + \tau(a)] - [\mu + \beta_2 + \tau(b)]$$

$$= 2(\beta_1 - \beta_2) + \tau(ab) + \tau(1) - \tau(a) - \tau(b).$$

Confounding example in 2^2 factorial experiment:

For example, in case *A* is confounded, then

- ***a* and *ab* with + signs are assigned to block 1 whereas**
- **(1) and *b* with – signs are assigned to block 2.**

Block 1	**Block 2**
a	**(1)**
ab	***b***

$$A = [\mu + \beta_1 + \tau(ab)] + [\mu + \beta_1 + \tau(a)] - [\mu + \beta_2 + \tau(b)] - [\mu + \beta_2 + \tau(1)]$$

$$= 2(\beta_1 - \beta_2) + \tau(ab) + \tau(a) - \tau(b) - \tau(1)$$

Confounding example in 2^2 factorial experiment:

The reason behind this observation is that if every block has treatment combinations in the form of linear contrast, then effects are estimable and thus unconfounded.

This is also evident from the theory of linear estimation that a linear parametric function is estimable if it is in the form of a linear contrast.

Confounding example in 2^2 factorial experiment:

The contrasts which are not estimable are said to be confounded with the differences between blocks (or block effects).

The contrasts which are estimable are said to be unconfounded with blocks or free from block effects.

Comparison of BIBD versus factorial:

Now we explain how confounding and BIBD compares together.

Consider a 2^3 factorial experiment which needs the block size to be 8.

Suppose the raw material available to conduct the experiment is sufficient only for a block of size 4. One can use BIBD in this case with parameters $b = 14, k = 4, v = 8, r = 7$ and $\lambda = 3$ (such BIBD exists).

For this BIBD, the efficiency factor is $E = \frac{\lambda v}{kr} = \frac{6}{8}$ and

$$Var(\hat{\tau}_j - \hat{\tau}_{j'})_{BIBD} = \frac{2k}{\lambda v}\sigma^2 = \frac{2}{6}\sigma^2 \quad (j \neq j').$$

Comparison of BIBD versus factorial:

Consider now an unconnected design in which 7 out of 14 blocks get treatment combination in block 1 as

$$\boxed{a \;\; b \;\; c \;\; abc}$$

and remaining 7 blocks get treatment combination in block 2 as

$$\boxed{(1) \;\; ab \;\; bc \;\; ac}$$

In this case, all the effects *A*, *B*, *C*, *AB*, *BC* and *AC* are estimable but *ABC* is not estimable because the treatment combinations with all + and all – signs in

$$\begin{aligned} ABC &= (a-1)(b-1)(c-1) \\ &= \underbrace{(a+b+c+abc)}_{\text{in block 1}} - \underbrace{((1)+ab+bc+ac)}_{\text{in block 2}} \end{aligned}$$

are contained in same blocks.

Comparison of BIBD versus factorial:

Note that in case of RBD,

$$Var(\hat{\tau}_j - \hat{\tau}_{j'})_{RBD} = \frac{2\sigma^2}{r} = \frac{2\sigma^2}{7} \quad (j \neq j')$$

and there are four linear contrasts, so the total variance is $4 \times (2\sigma^2 / 7)$ which gives the factor $8\sigma^2 / 7.$ and which is smaller than the variance under BIBD.

We observe that at the cost of not being able to estimate *ABC*, we have better estimates of *A*, *B*, *C*, *AB*, *BC* and *AC* with the same number of replicates as in BIBD.

Comparison of BIBD versus factorial:

Since higher order interactions are difficult to interpret and are usually not large, so it is much better to use confounding arrangements which provide better estimates of the interactions in which we are more interested.

Note that this example is for understanding only.

As such the concepts behind incomplete block design and confounding are different.

Confounding arrangement:

The arrangement of treatment combinations in different blocks, whereby some pre-determined effect (either main or interaction) contrasts are confounded is called a confounding arrangement.

For example, when the interaction *ABC* is confounded in a 2^3 factorial experiment, then the confounding arrangement consists of dividing the eight treatment combinations into following two sets:

$\boxed{a \;\; b \;\; c \;\; abc}$ **and** $\boxed{(1) \;\; ab \;\; bc \;\; ac}$

Confounding arrangement:

With the treatments of each set being assigned to the same block and each of these sets being replicated same number of times in the experiment, we say that we have a confounding arrangement of a 2^3 factorial in two blocks.

It may be noted that any confounding arrangement has to be such that only predetermined interactions are confounded and the estimates of interactions which are not confounded are orthogonal whenever the interactions are orthogonal.

Defining contrast and confounding arrangement:

The interactions which are confounded are called the defining contrasts of the confounding arrangement.

A confounded contrast will have treatment combinations with the same signs in each block of the confounding arrangement.

For example, if effect $AB = (a-1)(b-1)(c+1)$ **is to be confounded, then put all factor combinations with + sign, i.e., (1), *ab*, *c* and *abc* in one block and all other factor combinations with – sign, i.e., *a, b*, *ac* and *bc* in another block.**

So the block size reduces to 4 from 8 when one effect is confounded in 2^3 factorial experiment.

Confounding arrangement with two effects:

Suppose if along with *ABC* confounded, we want to confound *C* also.

To obtain such blocks, consider the blocks where *ABC* is confounded and divide them into further halves. So the block

$$\boxed{a \quad b \quad c \quad abc}$$

is divided into following two blocks: $\boxed{a \quad b}$ **and** $\boxed{c \quad abc}$

and the block

$$\boxed{(1) \quad ab \quad bc \quad ac}$$

is divided into following two blocks: $\boxed{(1) \quad ab}$ **and** $\boxed{bc \quad ac}$

Confounding arrangement with two effects:

These blocks of 4 treatments are divided into 2 blocks with each having 2 treatments and they are obtained in the following way.

If only *C* is confounded then the block with + sign of treatment combinations in *C* is

$$\boxed{c \quad ac \quad bc \quad abc}$$

and block with – sign of treatment combinations in *C* is

$$\boxed{(1) \quad a \quad b \quad ab}$$

Confounding arrangement with two effects:

Now look into the

(i) following block with + sign when $ABC = (a-1)(b-1)(c-1)$

is confounded, $\boxed{a \quad b \quad c \quad abc}$

(ii) following block with + sign when $C = (a+1)(b+1)(c-1)$

is confounded and $\boxed{c \quad ab \quad ac \quad abc}$

(iii) table of + and – signs in case of 2^3 factorial experiment.

Confounding arrangement with two effects:

Identify the treatment combinations having common - signs in these two blocks in (i) and (ii).

These treatment combinations are *c* and *abc.* So assign them into one block.

The remaining treatment combinations out of *a*, *b*, *c* and *abc* are *a* and *b* which go into another block.

Confounding arrangement with two effects:

Similarly look into the

(a) following block with – sign when ABC is confounded,

$$\boxed{(1) \;\; ab \;\; bc \;\; ac}$$

(b) following block with – sign when C is confounded and

$$\boxed{(1) \;\; a \;\; b \;\; ab}$$

(c) table of + and – signs in case of 2^3 factorial experiment.

Confounding arrangement with two effects:

Identify the treatment combinations having common – sign in these two blocks in (a) and (b).

These treatment combinations are (1) and *ab* which go into one block and the remaining two treatment combinations *ac* and *bc* out of *c, ac, bc* and *abc* go into another block.

So the blocks where both *ABC* and *C* are confounded together are

$$\boxed{(1)\quad ab},\ \boxed{a\quad b},\ \boxed{ac\quad bc}\ \text{ and }\ \boxed{c\quad abc}$$

Confounding arrangement with two effects:

While making these assignments of treatment combinations into four blocks, each of size two, we notice that another effect, viz., AB also gets confounded automatically.

Thus we see that when we confound two factors, a third factor is automatically getting confounded.

This situation is quite general.