Analysis of Variance and Design of Experiments

Confounding in 2^n Factorial Experiments

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Lecture 38

Definitions and Confounding Arrangement

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Defining contrast and confounding arrangement:

The interactions which are confounded are called the defining contrasts of the confounding arrangement.

A confounded contrast will have treatment combinations with the same signs in each block of the confounding arrangement.

Confounding arrangement:

The defining contrasts for a confounding arrangement cannot be chosen arbitrarily.

If some defining contrasts are selected then some other will also get confounded.

Now we present some definitions which are useful in describing the confounding arrangements.

Generalized interaction:

Given any two interactions, the generalized interaction is obtained by multiplying the factors (in capital letters) and ignoring all the terms with an even exponent.

For example, the generalized interaction of the factors *ABC* and *BCD* is

$$ABC \times BCD = AB^2C^2D = AD$$

and the generalized interaction of the factors *AB*, *BC* and *ABC* is

$$AB \times BC \times ABC = A^2B^3C^2 = B.$$

Independent set :

A set of main effects and interaction contrasts is called independent if no member of the set can be obtained as a generalized interaction of the other members of the set.

For example, the set of factors *AB*, *BC* and *AD* is an independent set but the set of factors *AB*, *BC*, *CD* and *AD* is not an independent set because

$$AB \times BC \times CD = AB^2C^2D = AD$$

which is already contained in the set.

Orthogonal treatment combinations:

The treatment combination $a^p b^q c^r$... is said to be orthogonal to the interaction $A^x B^y C^z ....$ if $(px + qy + rz + ...)$ is divisible by 2.

Since $p, q, r, ..., x, y, z, ...$ are either 0 or 1, so a treatment combination is orthogonal to an interaction if they have an even number of letters in common.

Treatment combination (1) is orthogonal to every interaction.

Orthogonal treatment combinations:

If $a^{p_1}b^{q_1}c^{r_1}....$**and** $a^{p_2}b^{q_2}c^{r_2}...$ **are both orthogonal to** $A^x B^y C^z....$ **then the product** $a^{p_1+p_2}b^{q_1+q_2}c^{r_1+r_2}...$ **is also orthogonal to** $A^x B^y C^z....$

Similarly, if two interactions are orthogonal to a treatment combination, then their generalized interaction is also orthogonal to it.

Results for confounding arrangements:

Now we give some general results for a confounding arrangement. Suppose we wish to have a confounding arrangement in 2^p blocks of a 2^k factorial experiment. Then we have the following observations:

1. The size of each block is 2^{k-p}
2. The number of elements in defining contrasts is $(2^p - 1)$, *i.e.,* $(2^p - 1)$ interactions have to be confounded.

Proof: If p factors are to be confounded, then the number of m^{th} order interaction with p factors is $\binom{p}{m}, \quad (m = 1, 2, ..., p).$

So the total number of factors to be confounded are $\sum_{m=1}^{p} \binom{p}{m} = 2^{p-1}.$

Results for confounding arrangements:

3. **If any two interactions are confounded, then their generalized interactions are also confounded.**
4. **The number of independent contrasts out of (2^p – 1) defining contrasts is *p* and rest are obtained as generalized interactions.**
5. **Number of effects getting confounded automatically is ($2^p - p - 1$) .**

Results for confounding arrangements:

To illustrate this, consider a 2^5 factorial ($k = 5$) with 5 factors, viz., *A*, *B*, *C*, *D* and *E*.

The factors are to be confounded in 2^3 blocks ($p = 3$).

So the size of each block is $2^{5-3} = 4$.

The number of defining contrasts is $2^3 - 1 = 7$

The number of independent contrasts which can be chosen arbitrarily is 3(i.e., *p*) out of 7 defining contrasts.

Results for confounding arrangements:

Suppose we choose *p* = 3 following independent contrasts as

(i) ***ACE***

(ii) ***CDE***

(iii) ***ABDE***

and then the remaining 4 out of 7 defining contrasts are obtained as

(iv) $(ACE)\times(CDE) = AC^2DE^2 = AD$

(v) $(ACE)\times(ABDE) = A^2BCDE^2 = BCD$

(vi) $(CDE)\times(ABDE) = ABCD^2E^2 = ABC$

(vii) $(ACE)\times(CDE)\times(ABDE) = A^2BC^2D^2E^3 = BE.$

Results for confounding arrangements:

Alternatively, if we choose another set of *p* = 3 independent contrast as

(i) ***ABCD*****,**

(ii) ***ACDE*****,**

(iii) ***ABCDE*****,**

then the defining contrasts are obtained as

(iv) $(ABCD)\times(ACDE) = A^2BC^2D^2E = BE$

(v) $(ABCD)\times(ABCDE) = A^2B^2C^2D^2E = E$

(vi) $(ACDE)\times(ABCDE) = A^2BC^2D^2E^2 = B$

(vii) $(ABCD)\times(ACDE)\times(ABCDE) = A^3B^2C^3D^3E^2 = ACD.$

In this case, the main effects *B* and *E* also get confounded.

Rules for confounding arrangements:

As a rule, try to confound, as far as possible, higher order interactions only because they are difficult to interpret.

After selecting *p* independent defining contrasts, divide the 2^k treatment combinations into 2^p groups of 2^{k-p} combinations each, and each group going into one block.

Group containing the combination (1) is called the principal block or key block. It contains all the treatment combinations which are orthogonal to the chosen independent defining contrasts.

Principal (key) block:

If there are *p* independent defining contrasts, then any treatment combination in principal block is orthogonal to *p* independent defining contrasts.

In order to obtain the principal block,

- write the treatment combinations in standard order.
- check each one of them for orthogonality.
- If two treatment combinations belongs to the principal block, their product also belongs to the principal block.
- When few treatment combinations of the principal block have been determined, other treatment combinations can be obtained by multiplication rule.

Principal (key) block: Example

Now we illustrate these steps in the following example.

Consider the set up of a 2^5 factorial experiment in which we want to divide the total treatment effects into 2^3 groups by confounding three effects *AD*, *BE* and *ABC*.

The generalized interactions in this case are *ADBE, BCD, ACE* are *CDE.*

Principal (key) block:

In order to find the principal block, first write the treatment combinations in standard order as follows:

(1)	*a*	*b*	*ab*	*c*	*ac*	*bc*	*abc*
d	*ad*	*bd*	*abd*	*cd*	*acd*	*bcd*	*abcd*
e	*ae*	*be*	*abe*	*ce*	*ace*	*bce*	*abce*
de	*ade*	*bde*	*abde*	*cde*	*acde*	*bcde*	*abcde*

Place a treatment combination in the principal block if it has an even number of letters in common with the confounded effects *AD, BE* and *ABC.*

The principal block has (1), *acd, bce* and *abde*(= *acd* x *bce*).

Principal (key) block:

Obtain other blocks of confounding arrangement from principal block by multiplying the treatment combinations of the principal block by a treatment combination not occurring in it or in any other block already obtained.

In other words, choose treatment combinations not occurring in it and multiply with them in the principal block.

Choose only distinct blocks.

Principal (key) block:

In this case, obtain other blocks by multiplying

a, b, ab, c, ac, bc, abc **like as in the following.**

Arrangement of the treatments in blocks when *AD, BE* and *ABC* are confounded

Principal Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8
(1)	a	b	ab	c	ac	bc	abc
acd bce $abde$	cd $abce$ bde	$abcd$ ce ade	bcd ace de	ad be $abcde$	d abe $bcde$	abd e $acde$	bd ae cde

Principal (key) block:

For example, block 2 is obtained by multiplying *a* with each factor combination in principal block as

$$(1)\times a = a,\ acd\times a = a^2cd = cd, bce\times a = abce,$$

$$abde\times a = a^2bde = bde;$$

block 3 is obtained by multiplying *b* with (1), *acd, bce* and *abde* and similarly other blocks are obtained.

If any other treatment combination is chosen to be multiplied with the treatments in principal block, then we get a block which will be one among the blocks 1 to 8.

Principal (key) block:

For example, if *ae* is multiplied with the treatments in principal block, then the blocks obtained consists of

$$(1)\times ae = ae,\ acd\times ae = cde,\ bce\times ae = abc$$

and which is same as the block 8.

Alternatively, if *ACD, ABCD* and *ABCDE* are to be confounded, then independent defining contrasts are *ACD, ABCD, ABCDE* and the principal block has (1), *ac, ad* and *cd* (= *ac* x *ad*).

Analysis of variance in case of confounded effects:

When an effect is confounded, it means that it is not estimable.

The following steps are followed to conduct the analysis of variance in case of factorial experiments with confounded effects:

- **Obtain the sum of squares due to main and interaction effects in the usual way as if no effect is confounded.**

- **Drop the sum of squares corresponding to confounded effects and retain only the sum of squares due to unconfounded effects.**

Analysis of variance in case of confounded effects:

- **Find the total sum of squares.**

- **Obtain the sum of squares due to error and associated degrees of freedom by subtraction.**

- **Conduct the test of hypothesis in the usual way.**