Analysis of Variance and Design of Experiments

Partial Confounding

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Lecture 39

Partial Confounding in 2^2 Factorial Experiment

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Partial confounding:

The objective of confounding is to mix the less important treatment combinations with the block effect differences so that higher accuracy can be provided to the other important treatment comparisons.

When such mixing of treatment contrasts and block differences is done in all the replicates, then it is termed as total confounding.

On the other hand, when the treatment contrast is not confounded in all the replicates but only in some of the replicates, then it is said to be partially confounded with the blocks.

Partial confounding:

It is also possible that one treatment combination is confounded in some of the replicates and another treatment combination is confounded in other replicates which are different from the earlier replicates.

So the treatment combinations confounded in some of the replicates and unconfounded in other replicates.

So the treatment combinations are said to be partially confounded.

Partial confounding:

The partially confounded contrasts are estimated only from those replicates where it is not confounded.

Since the variance of the contrast estimator is inversely proportional to the number of replicates in which they are estimable, so some factors on which information is available from all the replicates are more accurately determined.

Balanced and unbalanced partially confounded design:

If all the effects of a certain order are confounded with incomplete block differences in equal number of replicates in a design, then the design is said to be balanced partially confounded design.

If all the effects of a certain order are confounded an unequal number of times in a design, then the design is said to be unbalanced partially confounded design.

We discuss only the analysis of variance in the case of balanced partially confounded design through examples on 2^2 and 2^3 factorial experiments.

Partially confounded design in 2^2 factorial experiment:

Consider the case of 2^2 factorial as in following table in the set up of a randomized block design.

Factorial effects	Treatment combinations				Divisor
	(1)	*(a)*	*(b)*	*(ab)*	
M	+	+	+	+	4
A	−	+	−	+	2
B	−	−	+	+	2
AB	+	−	−	+	2

Let $y_{*i} = ((1), a, b, ab)'$ denotes the vector of total responses in the i^{th} replication and each treatment is replicated r times, $i = 1,2,\ldots,r$.

Partially confounded design in 2^2 factorial experiment:

If no factor is confounded then the factorial effects are estimated using all the replicates as

$$A = \frac{1}{2r}\sum_{i=1}^{r} \ell'_A y_{*i},$$

$$B = \frac{1}{2r}\sum_{i=1}^{r} \ell'_B y_{*i},$$

$$AB = \frac{1}{2r}\sum_{i=1}^{r} \ell'_{AB} y_{*i},$$

where the vectors of contrasts $\ell_A, \ell_B, \ell_{AB}$ **are given by**

$$\ell_A = (-1 \quad 1 \ -1 \quad 1)'$$

$$\ell_B = (-1 \ -1 \quad 1 \quad 1)'$$

$$\ell_{AB} = (1 \ -1 \ -1 \quad 1)'.$$

We have in this case $\ell'_A \ell_A = \ell'_B \ell_B = \ell'_{AB} \ell_{AB} = 4.$

Partially confounded design in 2^2 factorial experiment:

The sum of squares due to *A, B* and *AB* can be accordingly modified and expressed as

$$SS_A = \frac{(\sum_{i=1}^{r} \ell'_A y_{*i})^2}{r\ell'_A \ell_A} = \frac{(ab + a - b - (1))^2}{4r}$$

$$SS_B = \frac{(\sum_{i=1}^{r} \ell'_B y_{*i})^2}{r\ell'_B \ell_B} = \frac{(ab + b - a - (1))^2}{4r}$$

and

$$S_{AB} = \frac{(\sum_{i=1}^{r} \ell'_{AB} y_{*i})^2}{r\ell'_{AB} \ell_{AB}} = \frac{(ab + (1) - a - b)^2}{4r},$$

respectively.

Partially confounded design in 2^2 factorial experiment:

Now consider a situation with 3 replicates with each consisting of 2 incomplete blocks as in the following figure:

Replicate 1		Replicate 2		Replicate 3	
Block 1	Block 2	Block 1	Block 2	Block 1	Block 2
ab	b	ab	a	ab	a
a	(1)	b	(1)	(1)	b

There are three factors *A*, *B* and *AB* In case of total confounding, a factor is confounded in all the replicates.

We consider here the situation of partial confounding in which a factor is not confounded in all the replicates.

Partially confounded design in 2^2 factorial experiment:

Replicate 1		Replicate 2		Replicate 3	
(*A* confounded)		(*B* confounded)		(*AB* confounded)	
Block 1	Block 2	Block 1	Block 2	Block 1	Block 2
ab	b	ab	a	ab	a
a	(1)	b	(1)	(1)	b

Rather, the factor *A* is confounded in replicate 1,

the factor *B* is confounded in replicate 2 and

the interaction *AB* is confounded in replicate 3.

Suppose each of the three replicate is repeated *r* times.

So the observations are now available on *r* repetitions of each of the blocks in the three replicates.

Partially confounded design in 2^2 factorial experiment:

The partitions of replications, the blocks within replicates and plots within blocks being randomized.

Now from the setup of figure,

- **the factor A can be estimated from replicates 2 and 3 as it is confounded in replicate 1.**
- **the factor B can be estimated from replicates 1 and 3 as it is confounded in replicate 2 and**
- **the interaction AB can be estimated from replicates 1 and 2 as it is confounded in replicate 3.**

Partially confounded design in 2^2 factorial experiment:

When A is estimated from the replicate 2 only, then its estimate is given by

$$A_{rep2} = \frac{\left(\sum_{i=1}^{r} \ell'_{A2} y_{*i}\right)_{rep2}}{2r}$$

and when A is estimated from the replicate 3 only, then its estimate is given by

$$A_{rep3} = \frac{\left(\sum_{i=1}^{r} \ell'_{A3} y_{*i}\right)_{rep3}}{2r}$$

where ℓ'_{A2} and ℓ'_{A3} are the suitable vectors of +1 and -1 for being the linear function to be contrasts under replicates 2 and 3, respectively.

Partially confounded design in 2^2 factorial experiment:

Note that ℓ'_{A2} and ℓ'_{A3} each is a (4 x 1) vector having 4 elements in it. Now since *A* is estimated from both the replicates 2 and 3, so to combine them and to obtain a single estimator of *A*, we consider the arithmetic mean of A_{rep2} and A_{rep3} as an estimator of *A* given by

$$A_{pc} = \frac{A_{rep2} + A_{rep3}}{2} = \frac{(\sum_{i=1}^{r} \ell'_{A2} y_{*i})_{rep2} + (\sum_{i=1}^{r} \ell'_{A3} y_{*i})_{rep3}}{4r} = \frac{\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i}}{4r}$$

where (8 x 1) the vector $\ell^{*'}_{A} = (\ell_{A2}, \ell_{A3})$ has 8 elements in it and subscript *pc* in A_{pc} denotes the estimate of *A* under "partial confounding" (*pc*) .

Partially confounded design in 2^2 factorial experiment:

The sum of squares under partial confounding in this case is obtained as

$$SS_{Apc} = \frac{(\sum_{i=1}^{r} \ell_A^{*'} y_{*i})^2}{r \ell_A^{*'} \ell_A^{*}} = \frac{(\sum_{i=1}^{r} \ell_A^{*'} y_{*i})^2}{8r}$$

Assuming that y_{ij}'s are independent and $Var(y_{ij}) = \sigma^2$ for all *i* and *j*, the variance of A_{pc} is given by

$$\begin{aligned} Var(A_{pc}) &= \left(\frac{1}{4r}\right)^2 Var\left(\sum_{i=1}^{r} \ell_A^{*'} y_{*i}\right) \\ &= \left(\frac{1}{4r}\right)^2 Var\left((\sum_{i=1}^{r} \ell_{A2}^{'} y_{*i})_{rep2} + (\sum_{i=1}^{r} \ell_{A3}^{'} y_{*i})_{rep3}\right) \\ &= \left(\frac{1}{4r}\right)^2 \left(4r\sigma^2 + 4r\sigma^2\right) = \frac{\sigma^2}{2r}. \end{aligned}$$

Partially confounded design in 2^2 factorial experiment:

Now suppose *A* is not confounded in any of the blocks in the three replicates in this example.

Then *A* can be estimated from all the three replicates, each repeated *r* times.

Under such a condition, an estimate of *A* can be obtained using the same approach as the arithmetic mean of the estimates obtained from each of the three replicates as

$$A^*_{pc} = \frac{A_{rep1} + A_{rep2} + A_{rep3}}{3} = \frac{(\sum_{i=1}^{r} \ell'_{A1} y_{*i})_{rep1} + (\sum_{i=1}^{r} \ell'_{A2} y_{*i})_{rep2} + (\sum_{i=1}^{r} \ell'_{A3} y_{*i})_{rep3}}{6r} = \frac{\sum_{i=1}^{r} \ell^{**'}_{A} y_{*i}}{6r}$$

where (12 x 1) the vector $\ell^{**'}_{A} = \left(\ell_{A1},\ \ell_{A2},\ \ell_{A3}\right)$ **has 12 elements in it.**

Balanced and unbalanced partially confounded design

The variance of *A* assuming that y_{ij}*'s* are independent and

$Var(y_{ij}) = \sigma^{*2}$ **for all *i* and *j* in this case is obtained as**

$$Var(A^*_{pc}) = \left(\frac{1}{6r}\right)^2 Var\left[\left(\sum_{i=1}^{r} \ell'_{A1} y_{*i}\right)_{rep1} + \left(\sum_{i=1}^{r} \ell'_{A2} y_{*i}\right)_{rep2} + \left(\sum_{i=1}^{r} \ell'_{A3} y_{*i}\right)_{rep3}\right]$$

$$= \left(\frac{1}{6r}\right)^2 \left(4r\sigma^{*2} + 4r\sigma^{*2} + 4r\sigma^{*2}\right) = \frac{\sigma^{*2}}{3r}.$$

Partially confounded design in 2^2 factorial experiment:

If we compare this estimator with the earlier estimator for the situation where *A* is unconfounded in all the *r* replicates and was estimated by $A = \frac{\sum_{i=1}^{r} \ell'_A y_{*i}}{2r}$ **and in the present situation of partial confounding, the corresponding estimator of *A* is given by**

$$A_{pc}^{*} = \frac{A_{rep1} + A_{rep2} + A_{rep3}}{3} = \frac{\sum_{i=1}^{r} \ell_A^{**'} y_{*i}}{6r}$$

Partially confounded design in 2^2 factorial experiment:

Both the estimators, viz., A and A^*_{pc} are same because *A* is based on *r* replications whereas A^*_{pc} is based on 3*r* replications.

If we denote $r^* = 3r$ then A^*_{pc} becomes the same as *A*.

The expressions of variance of *A* and A^*_{pc} then also are same if we use $r^* = 3r.$

Comparing them, we see that the information on *A* in the partially confounded scheme relative to that in unconfounded scheme is $\dfrac{2r/\sigma^2}{3r/\sigma^{*2}} = \dfrac{2}{3}\dfrac{\sigma^{*2}}{\sigma^2}.$

If $\sigma^{*2} > \dfrac{3}{2}\sigma^2$, then the information in partially confounded design is more than the information in unconfounded design.

Partially confounded design in 2^2 factorial experiment:

Similarly, when *B* is estimated from the replicates 1 and 3 separately, then the individual estimates of *B* are given by

$$B_{rep1} = \frac{\left(\sum_{i=1}^{r} \ell'_{B1} y_{*i}\right)_{rep1}}{2r} \quad \textbf{and} \quad B_{rep3} = \frac{\left(\sum_{i=1}^{r} \ell'_{B3} y_{*i}\right)_{rep3}}{2r}.$$

Both the estimators are combined as arithmetic mean and the estimator of *B* based on partial confounding is

$$B_{pc} = \frac{B_{rep1} + B_{rep3}}{2} = \frac{\left(\sum_{i=1}^{r} \ell'_{B1} y_{*i}\right)_{rep1} + \left(\sum_{i=1}^{r} \ell'_{B3} y_{*i}\right)_{rep3}}{4r} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{B} y_{*i}\right)}{4r}$$

where the (8 x 1) vector $\ell^{*'}_{B} = (\ell_{B1},\ \ell_{B3})$ **as 8 elements.**

Partially confounded design in 2^2 factorial experiment:

The sum of squares due to B_{pc} is obtained as

$$SS_{B_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell_B^{*'} y_{*i}\right)^2}{r\ell_B^{*'}\ell_B^{*}} = \frac{\left(\sum_{i=1}^{r} \ell_B^{*'} y_{*i}\right)^2}{8r}.$$

Assuming that y_{ij}'s are independent and $Var(y_{ij}) = \sigma^2$, the variance of B_{pc} is

$$Var(B_{pc}) = \left(\frac{1}{4r}\right)^2 Var\left(\sum_{i=1}^{r} \ell_B^{*'} y_{*i}\right) = \frac{\sigma^2}{2r}.$$

Partially confounded design in 2^2 factorial experiment:

When *AB* is estimated from the replicates 1 and 2 separately, then its estimators based on the observations available from replicates 1 and 2 are

$$AB_{rep1} = \frac{\left(\sum_{i=1}^{r} \ell'_{AB1} y_{*i}\right)_{rep1}}{2r} \quad \text{and} \quad AB_{rep2} = \frac{\left(\sum_{i=1}^{r} \ell'_{AB2} y_{*1}\right)_{rep2}}{2r}$$

respectively.

Both the estimators are combined as arithmetic mean and the estimator of *AB* is obtained as

$$AB_{pc} = \frac{AB_{rep1} + AB_{rep2}}{2} = \frac{\left(\sum_{i=1}^{r} \ell'_{AB1} y_{*i}\right)_{rep1} + \left(\sum_{i=1}^{r} \ell'_{AB2} y_{*i}\right)_{rep2}}{4r} = \frac{\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}}{4r}$$

where (8 x 1) the vector $\ell^{*'}_{AB} = (\ell_{AB1}, \ \ell_{AB2})$ **consists of 8 elements.**

Partially confounded design in 2^2 factorial experiment:

The sum of squares due to AB_{pc} is

$$SS_{AB_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}\right)^2}{r\ell^{*}_{AB}\ell^{*}_{AB}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}\right)^2}{8r}$$

and the variance of AB_{pc} under the assumption that y_{ij}'s are independent and $Var\left(y_{ij}\right) = \sigma^2$ **is given by**

$$Var(AB_{pc}) = \left(\frac{1}{4r}\right)^2 Var\left(\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}\right)$$
$$= \frac{\sigma^2}{2r}.$$