Analysis of Variance and Design of Experiments

Partial Confounding

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Lecture 40

Partial Confounding in 2^2 and 2^3 Factorial Experiments

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Partially confounded design in 2^2 factorial experiment:

Replicate 1 (*A* confounded)		Replicate 2 (*B* confounded)		Replicate 3 (*AB* confounded)	
Block 1	**Block 2**	**Block 1**	**Block 2**	**Block 1**	**Block 2**
ab	b	ab	a	ab	a
a	(1)	b	(1)	(1)	b

The factor *A* is confounded in replicate 1,

the factor *B* is confounded in replicate 2 and

the interaction *AB* is confounded in replicate 3.

Suppose each of the three replicate is repeated *r* times.

So the observations are now available on *r* repetitions of each of the blocks in the three replicates.

Block sum of squares in 2^2 factorial experiment:

Note that is case of partial confounding, the block sum of squares will have two components – due to replicates and within replicates.

So the usual sum of squares due to blocks need to be divided into two components based on these two variants.

Now we illustrate how the sum of squares due to blocks are adjusted under partial confounding.

We consider the setup as in the earlier example. There are 6 blocks (2 blocks under each replicate 1, 2 and 3), each repeated r times.

Block sum of squares in 2^2 factorial experiment:

So there are total ($6r$ - 1) degrees of freedom associated with the sum of squares due to blocks. The sum of squares due to blocks is divided into two parts-

- the sum of squares due to replicates with ($3r$ - 1) degrees of freedom and
- the sum of squares due to within replicates with $3r$ degrees of freedom.

Now, denoting

- B_i to be the total of i^{th} block and
- R_i to be the total due to i^{th} replicate,

Block sum of squares in 2^2 factorial experiment:

$$SS_{Block(pc)} = \frac{1}{\text{Total number of treatment}} \sum_{i=1}^{\text{Total number of blocks}} B_i^2 - \frac{G^2}{N}$$

$$= \frac{1}{2^2}\sum_{i=1}^{3r} B_i^2 - \frac{G^2}{12r}; \ (N = 12r)$$

$$= \frac{1}{2^2}\sum_{i=1}^{3r} (B_i^2 - R_i^2 + R_i^2) - \frac{G^2}{12r}$$

$$= \frac{1}{2^2}\sum_{i=1}^{3r} \left(B_i^2 - R_i^2\right) + \left(\frac{1}{2^2}\sum_{i=1}^{3r} R_i^2 - \frac{G^2}{12r}\right)$$

$$= \frac{1}{2^2}\sum_{i=1}^{3r} \left(\frac{B_{1i}^2 + B_{2i}^2}{2} - R_i^2\right) + \left(\frac{1}{2^2}\sum_{i=1}^{3r} R_i^2 - \frac{G^2}{12r}\right)$$

where B_{ji} denotes the total of j^{th} block in i^{th} replicate (j = 1,2).

The sum of squares due to blocks within replications (*wr*) is

$$SS_{\text{Block}(wr)} = \frac{1}{2^2}\sum_{i=1}^{3r} \left(\frac{B_{1i}^2 + B_{2i}^2}{2} - R_i^2\right) \quad \text{and} \quad SS_{\text{Block}(r)} = \frac{1}{2^2}\sum_{i=1}^{3r} R_i^2 - \frac{G^2}{12r}$$

Block sum of squares in 2^2 factorial experiment:

So we have in case of partial confounding

$$SS_{\text{Block}} = SS_{\text{Block}(wr)} + SS_{\text{Block}(r)}$$

The total sum of squares remains the same as usual and is given by

$$SS_{\text{Total}(pc)} = \sum\sum\sum y_{ijk}^2 - \frac{G^2}{N}; \ (N = 12r).$$

ANOVA Table in 2^2 factorial experiment:

The analysis of variance table in this case of partial confounding is given in the following table.

The test of hypothesis can be carried out in a usual way as in the case of factorial experiments.

Source	Sum of squares	Degrees of freedom	Mean squares
Replicates	$SS_{\text{Block(r)}}$	$3r(=r^*)$	$MS_{\text{Block}}(r)$
Blocks within replicates	$SS_{\text{Block(wr)}}$	$3r-1(=r^*-1)$	$MS_{\text{Block(wr)}}$
Factor *A*	$SS_{A_{pc}}$	1	$MS_{A(pc)}$
Factor *B*	$SS_{B_{pc}}$	1	$MS_{B(pc)}$
AB	$SS_{AB_{pc}}$	1	$MS_{AB(pc)}$
Error	By subtraction	$6r-3(=2r^*-3)$	$MS_{E(pc)}$
Total	$SS_{\text{Total}(pc)}$	$12r-1(=4r^*-1)$	

Partially confounded design in 2^3 factorial experiment:

Consider the setup of 2^3 factorial experiment. The block size is 2^2 and 4 replications are made as in the following figure.

Replicate 1 (AB confounded)		Replicate 2 (AC confounded)		Replicate 3 (BC confounded)		Replicate 4 (ABC confounded)	
Block 1	Block 2	Block 1	Block 2	Block 1	Block 2	Block 1	Block 2
(1)	a	(1)	a	(1)	b	(1)	a
ab	b	b	ab	a	c	ab	b
c	ac	ac	c	bc	ab	ac	c
abc	bc	abc	bc	abc	ac	bc	abc

Partially confounded design in 2^3 factorial experiment:

The arrangement of the treatments in different blocks in various replicates is based on the fact that different interaction effects are confounded in the different replicates.

The interaction effect *AB* is confounded in replicate 1,

***AC* is confounded in replicate 2,**

***BC* is confounded in replicate 3 and**

***ABC* is confounded in replicate 4.**

Then the *r* replications of each block are obtained.

There are total eight factors involved in this case including (1).

Out of them, three factors, viz., *A*, *B* and *C* are unconfounded whereas *AB*, *BC*, *AC* and *ABC* are partially confounded.

Partially confounded design in 2^3 factorial experiment:

Our objective is to estimate all these factors. The unconfounded factors can be estimated from all the four replicates whereas partially confounded factors can be estimated from the following replicates:

- ***AB*** **from the replicates 2, 3 and 4,**
- ***AC*** **from the replicates 1, 3 and 4,**
- ***BC*** **from the replicates 1, 2 and 4 and**
- ***ABC*** **from the replicates 1, 2 and 3.**

Partially confounded design in 2^3 factorial experiment:

We first consider the estimation of unconfounded factors *A, B* and *C* which are estimated from all the four replicates.

The estimation of factor *A* from *j*th replicate (*j*=1,2,3,4) is as follows:

$$A_{rep\ j} = \frac{\sum_{i=1}^{r} \ell'_{Aj} y_{*i}}{4r}$$

$$A = \frac{\sum_{j=1}^{4} \ell'_{Aj} A_{rep\ j}}{4} = \frac{\sum_{j=1}^{4}\sum_{i=1}^{r} \ell'_{Aj} y_{*i}}{16r} = \frac{\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i}}{16r}$$

Partially confounded design in 2^3 factorial experiment:

where (32 x 1) the vector

$$\ell_A^{*'} = \left(\ell_{A1},\ \ell_{A2},\ \ell_{A3},\ \ell_{A4}\right)$$

has 32 elements and each $\ell_{Aj}\ (j = 1, 2, 3, 4)$ **is having 8 elements in it.**

The sum of square due to *A* is now based on 32 elements as

$$SS_A = \frac{\left(\sum_{i=1}^{r} \ell_A^{*'} y_{*i}\right)^2}{r\ell_A^{*'}\ell_A^{*}} = \frac{\left(\sum_{i=1}^{r} \ell_A^{*'} y_{*i}\right)^2}{32r}.$$

Partially confounded design in 2^3 factorial experiment:

Assuming that *y_{ij}'s* are independent and $Var(y_{ij}) = \sigma^2$ **for all *i* and *j*, the variance of *A* is obtained as**

$$Var(A) = \left(\frac{1}{16r}\right)^2 Var\left(\sum_{i=1}^{r} \ell_A^{*'} y_{*i}\right)$$

$$= \left(\frac{1}{16r}\right)^2 \times 32r\sigma^2$$

$$= \frac{\sigma^2}{8r},$$

Partially confounded design in 2^3 factorial experiment:

Similarly, the factor B is estimated as an arithmetic mean of the estimates of B from each replicate as

$$B = \frac{\sum_{i=1}^{r} \ell_{B}^{*'} y_{*i}}{16r}$$

where (32 x 1) the vector

$$\ell_{B}^{*'} = \left(\ell_{B1}, \ \ell_{B2}, \ \ell_{B3}, \ \ell_{B4} \right)$$

consists of 32 elements.

The sum of squares due to B is obtained on the similar line as in case of A as

$$SS_B = \frac{\left(\sum_{i=1}^{r} \ell_{B}^{*'} y_{*i} \right)^2}{32r}.$$

Partially confounded design in 2^3 factorial experiment:

The variance of *B* is obtained on the similar lines as in the case of *A* as

$$Var(B) = \frac{\sigma^2}{8r}$$

The unconfounded factor *C* is also estimated as the average of estimates of *C* from all the replicates as

$$C = \frac{\sum_{i=1}^{r} \ell_C^{*'} y_{*i}}{16r}$$

where (32 x 1) the vector $\ell_C^{*'} = \left(\ell_{C1},\ \ell_{C2},\ \ell_{C3},\ \ell_{C4}\right)$

consists of 32 elements.

Partially confounded design in 2^3 factorial experiment:

The sum of square due to *C* in this case is obtained as

$$SS_C = \frac{\left(\sum_{i=1}^{r} \ell_C^{*'} y_{*i}\right)^2}{32r}.$$

The variance of *C* is obtained as $Var(C) = \frac{\sigma^2}{8r}.$

Partially confounded design in 2^3 factorial experiment:

Next we consider the estimation of the confounded factor *AB*.

This factor *AB* can be estimated from each of the replicates 2, 3 and 4 and the final estimate of *AB* can be obtained as the arithmetic mean of those three estimates as

$$AB_{pc} = \frac{AB_{rep2} + AB_{rep3} + AB_{rep4}}{3}$$

$$= \frac{1}{12r}\left(\left(\sum_{i=1}^{r} \ell'_{AB2} y_{*i})_{rep2}\right) + \left(\sum_{i=1}^{r} \ell'_{AB3} y_{*i})_{rep3}\right) + \left(\sum_{i=1}^{r} \ell'_{AB3} y_{*i})_{rep4}\right)\right)$$

$$= \frac{\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}}{12r}$$

where (24 x 1) the vector $\ell^{*'}_{AB} = \left(\ell_{AB2}, \ \ell_{AB3}, \ \ell_{AB4}\right)$

consists of 24 elements and each of the (8 x 1) vectors ℓ_{AB2}, ℓ_{AB3} **and** ℓ_{AB4} **is having 8 elements in it.**

Partially confounded design in 2^3 factorial experiment:

The sum of squares due to *AB_{pc}* is then based on 24 elements given as

$$SS_{AB_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell'^{*}_{AB} y_{*i}\right)^2}{r\ell'^{*}_{AB}\ell^{*}_{AB}} = \frac{\left(\sum_{i=1}^{r} \ell'^{*}_{AB} y_{*i}\right)^2}{24r}.$$

Partially confounded design in 2^3 factorial experiment:

The variance AB_{pc} **in this case is obtained under the assumption that** y_{ij}**'s are independent and each has variance** σ^2 **as**

$$
\begin{aligned}
Var(AB_{pc}) &= \left(\frac{1}{12r}\right)^2 Var\left(\sum_{i=1}^{r} \ell^{*'}_{AB} y_{*i}\right) \\
&= \left(\frac{1}{12r}\right)^2 Var\left(\left(\sum_{i=1}^{r} \ell^{*'}_{AB2} y_{*i}\right)_{rep2} + \left(\sum_{i=1}^{r} \ell^{*'}_{AB3} y_{*i}\right)_{rep3} + \left(\sum_{i=1}^{r} \ell^{*'}_{AB4} y_{*i}\right)_{rep4}\right) \\
&= \left(\frac{1}{12r}\right)^2 (8r\sigma^2 + 8r\sigma^2 + 8r\sigma^2) \\
&= \frac{\sigma^2}{6r}.
\end{aligned}
$$

Partially confounded design in 2^3 factorial experiment:

The confounded effects *AC* is obtained as the average of estimates of *AC* obtained from the replicates 1, 3 and 4 as

$$AC_{pc} = \frac{AC_{rep1} + AC_{rep3} + AC_{rep4}}{3}$$

$$= \frac{\sum_{i=1}^{r} \ell^{*'}_{AC} y_{*i}}{12r}$$

where (24 x 1) the vector $\ell'^{*}_{AC} = (\ell_{AC1}, \ \ell_{AC3}, \ \ell_{AC4})$ **consists of 24 elements,**

The sum of squares due to AC_{pc} **in this case is given by**

$$SS_{AC_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell'^{*}_{AC} y_{*i} \right)^2}{24r}.$$

Partially confounded design in 2^3 factorial experiment:

The variance of *AC* in this case under the assumption that y_{ij}'s are independent and each has variance σ^2 is given by

$$Var(AC_{pc}) = \frac{\sigma^2}{6r}.$$

Partially confounded design in 2^3 factorial experiment:

Similarly, the confounded effect *BC* is estimated as the average of the estimates of *BC* obtained from the replicates 1, 2 and 4 as

$$BC_{pc} = \frac{BC_{rep1} + BC_{rep2} + BC_{rep4}}{3}$$

$$= \frac{\sum_{i=1}^{r} \ell'^{*}_{BC} y_{*i}}{12r}$$

Partially confounded design in 2^3 factorial experiment:

where the vector

$$\ell^{*'}_{BC} = (\ell_{BC1}, \ \ell_{BC3}, \ \ell_{BC4})$$

consists of 24 elements.

The sum of squares due to *BC* in this case is based on 24*r* elements and is given as

$$SS_{BC_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{BC} y_{*i}\right)^2}{24r}.$$

The variance of *BC* in this case is obtained under the assumption that y_{ij}*'s* are independent and each has variance σ^2 as

$$Var(BC_{pc}) = \frac{\sigma^2}{6r}$$

Partially confounded design in 2^3 factorial experiment:

Lastly, the confounded effect *ABC* can be estimated first from the replicates 1,2 and 3 and then the estimate of *ABC* obtained as an average of these three individual estimates as

$$ABC_{pc} = \frac{ABC_{rep1} + ABC_{rep2} + ABC_{rep3}}{3} = \frac{\sum_{i=1}^{r} \ell'^{*}_{ABC} y_{*i}}{12r}$$

where (24 x 1) the vector

$$\ell^{*'}_{ABC} = \left(\ell_{ABC1},\ \ell_{ABC2},\ \ell_{ABC3} \right)$$

consists of 24 elements,

The sum of squares due to *ABC* in this case is based on 24*r* elements and is given by

$$SS_{ABC_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{ABC} y_{*i} \right)^2}{24r}.$$

Partially confounded design in 2^3 factorial experiment:

The variance of *ABC* in this case assuming that y_{ij}'s are independent and each has variance σ^2 is given by

$$Var(ABC_{pc}) = \frac{\sigma^2}{6r}.$$

Partially confounded design in 2^3 factorial experiment:

If an unconfounded design with *4r* replication was used then the variance of each of the factors *A, B, C, AB, BC, AC* and *ABC* is $\sigma^{*2}/8r$ where σ^{*2} is the error variance on blocks of size 8. So the relative efficiency of a confounded effect in the partially confounded design with respect to that of an unconfounded one in a comparable design is

$$\frac{6r/\sigma^2}{8r/\sigma^{*2}} = \frac{3}{4}\frac{\sigma^{*2}}{\sigma^2}.$$

Block sum of squares:

So the information on a partially confounded effect relative to an unconfounded effect is $\frac{3}{4}$. If $\sigma^{*2} > 4\sigma^2/3$, then partially confounded design gives more information than the unconfounded design.

Further, the sum of squares due to block can be divided into two components – within replicates and due to replications. So we can write

$$SS_{\text{Block}} = SS_{\text{Block(wr)}} + SS_{\text{Block(r)}}$$

where the sum of squares due to blocks within replications (*wr*) is

$$SS_{\text{Block(wr)}} = \frac{1}{2^3}\sum_{i=1}^{4r}\left(\frac{B_{1i}^2 + B_{2i}^2}{2} - R_i^2\right)$$

Block sum of squares:

which carries *4r* degrees of freedom and the sum of squares due to replication is

$$SS_{\text{Block(r)}} = \frac{1}{2^3}\sum_{i=1}^{4r} R_i^2 - \frac{G^2}{32r}$$

which carries (*4r - 1*) degrees of freedom. The total sum of squares is

$$SS_{\text{Total(pc)}} = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{G^2}{32r}$$

The analysis of variance table in this case of 2^3 factorial under partial confounding is given as

ANOVA table:

Source	Sum of squares	Degrees of freedom	Mean squares
Replicates	$SS_{Block(r)}$	4r - 1	$MS_{Block(r)}$
Blocks within replicate	$SS_{Block(wr)}$	4r	$MS_{Block(wr)}$
Factor *A*	SS_A	1	MS_A
Factor *B*	SS_B	1	MS_B
Factor C	SS_C	1	MS_C
AB	$SS_{AB(pc)}$	1	$MS_{AB(pc)}$
AC	$SS_{AC(pc)}$	1	$MS_{AC(pc)}$
BC	$SS_{BC(pc)}$	1	$MS_{BC(pc)}$
ABC	$SS_{ABC(pc)}$	1	$MS_{ABC(pc)}$
Error	by subtraction	$24r$ - 7	$MS_{E(pc)}$
Total	$SS_{Total(pc)}$	$32r - 1$	

Test of hypothesis can be carried out in the usual way as in the case of factorial experiment.