Analysis of Variance and Design of Experiments

General Linear Hypothesis and Analysis of Variance

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Lecture 6

Test of Hypothesis for Equality of Parameters

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Tests of Hypothesis in the Linear Regression Model

First, we discuss the development of the tests of hypothesis concerning the parameters of a linear regression model. These tests of the hypothesis will be used later in the development of tests based on the analysis of variance.

Analysis of Variance

The technique in the analysis of variance involves the breaking down of total variation into orthogonal components.

Each orthogonal factor represents the variation due to a particular factor contributing in the total variation.

Tests of Hypothesis in the Linear Regression Model

Model

Let $Y_1, Y_2, \ldots, Y_n$ be independently distributed following a normal distribution with mean $E(Y_i) = \sum_{j=1}^{p} \beta_j x_{ij}$ and variance σ^2.

Denoting $Y = (Y_1, Y_2, \ldots, Y_n)'$ a *n* x 1 column vector, such assumption can be expressed in the form of a linear regression model

$$Y = X\beta + \varepsilon$$

where *X* is a *n* x *p* matrix, β is a *p* x 1 vector and ε is a *n* x 1 vector of disturbances with $E(\varepsilon) = 0, Cov(\varepsilon) = \sigma^2 I$ and ε follows a normal distribution. This implies that

$$E(Y) = X\beta, \qquad Var(Y) = E(Y - X\beta)(Y - X\beta)' = \sigma^2 I.$$

Tests of Hypothesis in the Linear Regression Model

We develop the likelihood ratio test for the null hypothesis related to the analysis of variance.

Note that, later we will derive the same test on the basis of least squares principle also.

An important idea behind the development of this test is to demonstrate that the test used in the analysis of variance can be derived using the least-squares principle as well as the likelihood ratio test.

Tests of Hypothesis in the LRM: H_0: $\beta=\beta_0$

Consider the null hypothesis for testing $H_0 : \beta = \beta^0$ **where**

$\beta = (\beta_1, \beta_2, ..., \beta_p)', \beta^0 = (\beta_1^0, \beta_2^0, ..., \beta_p^0)'$ **is specified and** σ^2 **is unknown.**

This null hypothesis is equivalent to $H_0 : \beta_1 = \beta_1^0, \beta_2 = \beta_2^0, ..., \beta_p = \beta_p^0.$

Assume that all β_i**'s are estimable, i.e.,** ***rank*(*X*) = *p*** **(full column rank).**

We now develop the likelihood ratio test.

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

The (*p* + 1) x 1 dimensional parametric space Ω is a collection of points such that

$$\Omega = \left\{(\beta, \sigma^2);\ -\infty < \beta_i < \infty,\ \sigma^2 > 0, i = 1, 2, \ldots p\right\}.$$

Under H_0, all β's are known and equal, say β^0 all are known and the Ω reduces to one-dimensional space given by

$$\omega = \left\{(\beta^0, \sigma^2);\ \sigma^2 > 0\right\}$$

The likelihood function of $y_1, y_2, \ldots, y_n$ is

$$L(y|\beta, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta^0)'(y - X\beta^0)\right].$$

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

The likelihood function is maximum over Ω when β and σ^2 are substituted with their maximum likelihood estimators, i.e.,

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$\hat{\sigma}^2 = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta}).$$

Substituting $\hat{\beta}$ and $\hat{\sigma}^2$ in $L(y \mid \beta, \sigma^2)$ gives

$$\underset{\Omega}{Max}\ L(y \mid \beta, \sigma^2) = \left(\frac{1}{2\pi\hat{\sigma}^2}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2\hat{\sigma}^2}(y - X\hat{\beta})'(y - X\hat{\beta})\right)$$

$$= \left(\frac{n}{2\pi(y - X\hat{\beta})'(y - X\hat{\beta})}\right)^{\frac{n}{2}} \exp\left(-\frac{n}{2}\right).$$

Tests of Hypothesis in the LRM: H_0: $\beta=\beta_0$

Under H_0, the maximum likelihood estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n}(y - X\beta^0)'(y - X\beta^0).$$

The maximum value of the likelihood function under H_0 is

$$\underset{\omega}{Max}\ L(y|\beta,\sigma^2) = \left(\frac{1}{2\pi\hat{\sigma}^2}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2\hat{\sigma}^2}(y - X\beta^0)'(y - X\beta^0)\right)$$

$$= \left(\frac{n}{2\pi(y - X\beta^0)'(y - X\beta^0)}\right)^{\frac{n}{2}} \exp\left(-\frac{n}{2}\right)$$

Tests of Hypothesis in the LRM: H_0: $\beta=\beta_0$

The likelihood ratio test statistic is

$$\lambda = \frac{\underset{\omega}{Max}\, L(y|\beta,\sigma^2)}{\underset{\Omega}{Max}\, L(y|\beta,\sigma^2)}$$

$$= \left[\frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{(y-X\beta^0)'(y-X\beta^0)}\right]^{\frac{n}{2}}$$

$$= \left[\frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{\left[(y-X\hat{\beta})+(X\hat{\beta}-X\beta^0)\right]'\left[(y-X\hat{\beta})+(X\hat{\beta}-X\beta^0)\right]}\right]^{\frac{n}{2}}$$

$$= \left[1+\frac{(\hat{\beta}-\beta^0)'X'X(\hat{\beta}-\beta^0)}{(y-X\hat{\beta})'(y-X\hat{\beta})}\right]^{-\frac{n}{2}} = \left[1+\frac{q_1}{q_2}\right]^{-\frac{n}{2}}$$

where $q_1 = (\hat{\beta}-\beta^0)'X'X(\hat{\beta}-\beta^0)$ **and** $q_2 = (y-X\hat{\beta})'(y-X\hat{\beta})$.

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

The expression of q_1 and q_2 can be further simplified as follows.

Consider

$$q_1 = (\hat{\beta} - \beta^0)' X'X (\hat{\beta} - \beta^0)$$

$$= \left[(X'X)^{-1} X'y - \beta^0 \right]' X'X \left[(X'X)^{-1} X'y - \beta^0 \right]$$

$$= \left[(X'X)^{-1} X'(y - X\beta^0) \right]' X'X \left[(X'X)^{-1} X'(y - X\beta^0) \right]$$

$$= (y - X\beta^0)' X (X'X)^{-1} X'X (X'X)^{-1} X'(y - X\beta^0)$$

$$= (y - X\beta^0)' X (X'X)^{-1} X'(y - X\beta^0)$$

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

$$q_2 = (y - X\hat{\beta})'(y - X\hat{\beta})$$

$$= \left[y - X(X'X)^{-1}X'y \right]' \left[y - X(X'X)^{-1}X'y \right]$$

$$= y'\left[I - X(X'X)^{-1}X' \right] y$$

$$= [(y - X\beta^0) + X\beta^0]'[I - X(X'X)^{-1}X'][(y - X\beta^0)' + X\beta^0]$$

$$= (y - X\beta^0)'[I - X(X'X)^{-1}X'](y - X\beta^0)$$

Other two terms become zero using

$$[I - X(X'X)^{-1}X']X = 0$$

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

In order to find out the decision rule for H_0 based on λ, first, we need to find if λ is a monotonic increasing or decreasing function of $\frac{q_1}{q_2}$. So we proceed as follows:

Let $g = \frac{q_1}{q_2}$, **so that** $\lambda = \left(1 + \frac{q_1}{q_2}\right)^{-\frac{n}{2}} = (1+g)^{-\frac{n}{2}}$,

then $\frac{d\lambda}{dg} = -\frac{n}{2}\frac{1}{(1+g)^{\frac{n}{2}+1}}$.

So as *g* increases, λ decreases. Thus λ is a monotonic decreasing function of $\frac{q_1}{q_2}$.

Decision rule: **Reject H_0 if $\lambda \le \lambda_0$ where λ_0 is a constant to be determined on the basis of size of the test α.**

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

Let us simplify this in our context.

$$\lambda \le \lambda_0$$

or $$\left(1 + \frac{q_1}{q_2}\right)^{-\frac{n}{2}} \le \lambda_o$$

or $$\frac{1}{(1+g)^{\frac{n}{2}}} \le \lambda_o$$

or $$(1+g) \ge \lambda_0^{-\frac{2}{n}}$$

or $$g \ge \lambda_0^{-\frac{2}{n}} - 1$$

or $$g \ge C$$

where *C* is a constant to be determined by the size α condition of the test. So reject H_0 whenever $\dfrac{q_1}{q_2} \ge C.$

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

Note that the statistic $\frac{q_1}{q_2}$ can also be obtained by the least-squares method.

The least-squares methodology will also be discussed in further lectures.

It will be seen later that the test statistic will be based on the ratio $\frac{q_1}{q_2}$.

In order to find an appropriate distribution of $\frac{q_1}{q_2}$, we use the following theorem:

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

Theorem 9: Let $Z = Y - X\beta^0$

$$Q_1 = Z'X(X'X)^{-1}X'Z$$

$$Q_2 = Z'[I - X(X'X)^{-1}X']Z.$$

Then $\frac{Q_1}{\sigma^2}$ **and** $\frac{Q_2}{\sigma^2}$ **are independently distributed.**

Further, when H_0 **is true, then**

$$\frac{Q_1}{\sigma^2} \sim \chi^2(p)$$

and $\frac{Q_2}{\sigma^2} \sim \chi^2(n-p)$

where $\chi^2(m)$ **denotes the** χ^2 **distribution with '*m*' degrees of freedom.**

Tests of Hypothesis in the LRM: H_0: $\beta=\beta_0$

Proof of Theorem 9: **Under H_0,**

$$E(Z) = X\beta^0 - X\beta^0 = 0$$

$$Var(Z) = Var(Y) = \sigma^2 I.$$

Further Z is a linear function of Y and Y follows a normal distribution. So $Z \sim N(0, \sigma^2 I)$

The matrices $X(X'X)^{-1}X'$ **and** $[I - X(X'X)^{-1}X']$ **are idempotent matrices. So**

$$tr[X(X'X)^{-1}X'] = tr[(X'X)^{-1}X'X] = tr(I_p) = p$$

$$tr[I - X(X'X)^{-1}X'] = tr\, I_n - tr[X(X'X)^{-1}X'] = n - p$$

Tests of Hypothesis in the LRM: H_0: $\beta=\beta_0$

Proof of Theorem 9: **So using theorem 6, we can write that under *H_0***

$$\frac{Q_1}{\sigma^2} \sim \chi^2(p) \quad \text{and} \quad \frac{Q_2}{\sigma^2} \sim \chi^2(n-p)$$

where the degrees of freedom *p* and *(n - p)* are obtained by the trace of $\left[X(X'X)^{-1}X'\right]$ and trace of $\left[I - X(X'X)^{-1}X'\right]$, respectively.

Since $\left[I - X(X'X)^{-1}X'\right]\left[X(X'X)^{-1}X'\right] = 0,$

so using theorem 7, the quadratic forms Q_1 and Q_2 are independent under H_0.

Hence the theorem is proved.

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

Since Q_1 and Q_2 are independently distributed, so under H_0

$\dfrac{Q_1 / p}{Q_2 /(n-p)}$ **follows a central *F* distribution, i.e.**

$$\left(\frac{n-p}{p}\right)\frac{Q_1}{Q_2} \sim F(p, n-p).$$

Hence the constant *C* in the likelihood ratio test statistic λ is given by

$$C = F_{1-\alpha}(p, n-p)$$

where $F_{1-\alpha}(n_1, n_2)$ denotes the upper $100\alpha\%$ points of *F*-distribution with n_1 and n_2 degrees of freedom.

Tests of Hypothesis in the LRM: H_0: $\beta = \beta_0$

The computations of this test of hypothesis can be represented in the form of an analysis of variance table.

ANOVA for testing H_0: $\beta = \beta_0$

Source of variation	Degrees of freedom	Sum of squares	Mean squares	*F*-value
Due to β	p	q_1	$\frac{q_1}{p}$	$\left(\frac{n-p}{p}\right)\frac{q_1}{q_2}$
Error	$n - p$	q_2	$\frac{q_2}{(n-p)}$	
Total	n	$(y - X\beta^0)'(y - X\beta^0)$		