Analysis of Variance and Design of Experiments

General Linear Hypothesis and Analysis of Variance

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Lecture 7

Test of Hypothesis for Linear Parametric Functions

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Tests of Hypothesis in the Linear Regression Model

Model

Denoting $Y = (Y_1, Y_2, \ldots, Y_n)'$ **a *n* x 1 column vector, such assumption can be expressed in the form of a linear regression model**

$$Y = X\beta + \varepsilon$$

where *X* is a *n* x *p* matrix, β **is a *p* x 1 vector and** ε **is a *n* x 1 vector of disturbances with** $E(\varepsilon) = 0, Cov(\varepsilon) = \sigma^2 I$ **and** ε **follows a normal distribution.**

This implies that

$$E(Y) = X\beta, \qquad Var(Y) = E(Y - X\beta)(Y - X\beta)' = \sigma^2 I.$$

Tests of Hypothesis in the LRM: H_0: $\beta = \beta^0$

ANOVA for testing H_0: $\beta = \beta^0$

Source of variation	Degrees of freedom	Sum of squares	Mean squares	*F*-value
Due to β	p	q_1	$\frac{q_1}{p}$	$\left(\frac{n-p}{p}\right)\frac{q_1}{q_2}$
Error	$n - p$	q_2	$\frac{q_2}{(n-p)}$	
Total	n	$(y - X\beta^0)'(y - X\beta^0)$		

$$q_1 = (y - X\beta^0)' X(X'X)^{-1} X'(y - X\beta^0)$$

$$q_2 = (y - X\beta^0)'[I - X(X'X)^{-1} X'](y - X\beta^0)$$

If $F > F_{1-\alpha}(p-1, n-p)$, **then** $H_0 : \beta_1 = \beta_2 = \ldots = \beta_p$ **is rejected.**

Tests of Hypothesis in the LRM: H_0: $L'\beta=\delta$

Let us consider the test of hypothesis related to a linear parametric function.

Assuming that the linear parameter function $L'\beta$ **is estimable where** $L = (\ell_1, \ell_2, ..., \ell_p)'$ **is a *p* x 1 vector of known constants and** $\beta = (\beta_1, \beta_2, ..., \beta_p)'$**.**

The null hypothesis of interest is $H_0 : L'\beta = \delta$ **where** δ **is some specified constant.**

Tests of Hypothesis in the LRM: $H_0: L'\beta=\delta$

Consider the set up of linear model $Y = X\beta + \varepsilon$ **where**

$Y = (Y_1, Y_2, ..., Y_n)'$ **follows** $N(X\beta, \sigma^2 I).$

The maximum likelihood estimators of β **and** σ^2 **are**

$$\hat{\beta} = (X'X)^{-1}X'y$$

and

$$\hat{\sigma}^2 = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta})$$

respectively.

Tests of Hypothesis in the LRM: H_0: $L'\beta=\delta$

The maximum likelihood estimate of estimable $L'\beta$ is $L'\hat{\beta}$, with

$$E(L'\hat{\beta}) = L'\beta$$

$$Cov(L'\hat{\beta}) = \sigma^2 L'(X'X)^{-1} L$$

$$L'\hat{\beta} \sim N\left[L'\beta, \sigma^2 L'(X'X)^{-1} L\right]$$

and $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-p)$ **assuming *X* to be the full column rank matrix.**

Further, $L'\hat{\beta}$ **and** $\frac{n\hat{\sigma}^2}{\sigma^2}$ **are also independently distributed.**

Tests of Hypothesis in the LRM: H_0: $L'\beta=\delta$

Under $H_0 : L'\beta = \delta,$ **the statistic**

$$t = \frac{\sqrt{(n-p)}(L'\hat{\beta}-\delta)}{\sqrt{n\hat{\sigma}^2 L'(X'X)^{-1}L}}$$

follows a *t*-distribution with (*n - p*) degrees of freedom.

So the test for $H_0 : L'\beta = \delta$ **against** $H_1 : L'\beta \neq \delta$ **rejects** H_0 **whenever**

$$|t| \geq t_{1-\frac{\alpha}{2}}(n-p)$$

where $t_{1-\alpha}(n_1)$ **denotes the upper 100**α **% points on *t*-distribution with** n_1 **degrees of freedom.**

Tests of Hypothesis in the LRM: H_0: $\phi_1 = \delta_1, \phi_2 = \delta_2, \ldots, \phi_k = \delta_k$

Now we develop the test of hypothesis related to more than one linear parametric functions.

Let the i^{th} **estimable linear parametric function is** $\phi_i = L_i^{'}\beta$ **and there are** k **such functions with** L_i **and** β **both being** p **x 1 vectors.**

Our interest is to test the hypothesis

$$H_0 : \phi_1 = \delta_1, \phi_2 = \delta_2, \ldots, \phi_k = \delta_k$$

where $\delta_1, \delta_2, \ldots, \delta_k$ **are the known constants.**

Tests of Hypothesis in the LRM: H_0: $\phi_1 = \delta_1, \phi_2 = \delta_2, \ldots, \phi_k = \delta_k$

Let $\phi = (\phi_1, \phi_2, \ldots, \phi_k)'$ **and** $\delta = (\delta_1, \delta_2, .., \delta_k)'$.

Then H_0 **is expressible as** $H_0 : \phi = L'\beta = \delta$

where L' **is a** k **x** p **matrix of constants associated with** $L_1, L_2, \ldots, L_k$.

The maximum likelihood estimator of ϕ_i **is :** $\hat{\phi}_i = L_i' \hat{\beta}$

where $\hat{\beta} = (X'X)^{-1} X'y$.

Then $\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_k)' = L'\hat{\beta}$.

Tests of Hypothesis in the LRM: H_0: $\phi_1 = \delta_1, \phi_2 = \delta_2, \ldots, \phi_k = \delta_k$

Also $E(\hat{\phi}) = \phi$ **and** $Cov(\hat{\phi}) = \sigma^2 V$

where $V = ((L_i'(X'X)^{-1}L_j))$ **,** $(L_i'(X'X)^{-1}L_j)$ **is the** $(i, j)^{th}$ **element of** ***V*****.**

Thus

$$\frac{(\hat{\phi}-\phi)'V^{-1}(\hat{\phi}-\phi)}{\sigma^2}$$

follows a $\chi^2 -$ **distribution with *k* degrees of freedom and**

$\dfrac{n\hat{\sigma}^2}{\sigma^2}$ **follows a** $\chi^2 -$ **distribution with (*n* - *p*) degrees of freedom**

where $\hat{\sigma}^2 = \dfrac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta})$ **is the maximum likelihood estimator of** σ^2**.**

Tests of Hypothesis in the LRM: H_0: $\phi_1 = \delta_1, \phi_2 = \delta_2, \ldots, \phi_k = \delta_k$

Further $\dfrac{(\hat{\phi}-\phi)'V^{-1}(\hat{\phi}-\phi)}{\sigma^2}$ **and** $\dfrac{n\hat{\sigma}^2}{\sigma^2}$ **are also independently distributed.**

Thus under $H_0 : \phi = \delta$

$$\frac{\left(\dfrac{\dfrac{(\hat{\phi}-\delta)'V^{-1}(\hat{\phi}-\delta)}{\sigma^2}}{k}\right)}{\left(\dfrac{\dfrac{n\hat{\sigma}^2}{\sigma^2}}{(n-p)}\right)} \sim F(k, n-p)$$

or $$\left(\frac{n-p}{k}\right)\frac{(\hat{\phi}-\delta)'V^{-1}(\hat{\phi}-\delta)}{n\hat{\sigma}^2} \sim F(k, n-p)$$

Tests of Hypothesis in the LRM: H_0: $\phi_1 = \delta_1, \phi_2 = \delta_2, \ldots, \phi_k = \delta_k$

So the hypothesis $H_0 : \phi = \delta$ **is rejected against**

***H*$_1$: At lest one** $\phi_i \neq \delta_i$ **for** $i = 1, 2, \ldots, k$ **whenever**

$$F \geq F_{1-\alpha}(k, n-p)$$

where $F_{1-\alpha}(k, n-p)$ **denotes the** $100\alpha\%$ **points on *F*-distribution with *k* and (*n* – *p*) degrees of freedom.**

One-way classification with fixed effect linear models of full rank:

The objective in the one-way classification is to test the hypothesis about the equality of means on the basis of several samples which have been drawn from univariate normal populations with different means but the same variances.

Let there be *p* univariate normal populations and samples of different sizes are drawn from each of the population.

One-way classification with fixed effect linear models of full rank:

Let y_{ij}(j = 1,2,…, n_i) be a random sample from the i^{th} normal population with mean β_i and variance $\sigma^2, i = 1, 2, ..., p$, i.e.,

$$Y_{ij} \sim N(\beta_i, \sigma^2), \; j = 1, 2, ..., n_i; \; i = 1, 2, ..., p.$$

The random samples from different populations are assumed to be independent of each other.

These observations follow the set up of linear model

$$Y = X\beta + \varepsilon$$

One-way classification with fixed effect linear models of full rank:

These observations follow the set up of linear model

$$Y = X\beta + \varepsilon$$

where

$$Y = (Y_{11}, Y_{12}, \ldots, Y_{1n_1}, Y_{21}, \ldots, Y_{2n_2}, \ldots, Y_{p1}, Y_{p2}, \ldots, Y_{pn_p})'$$

$$y = (y_{11}, y_{12}, \ldots, y_{1n_1}, y_{21}, \ldots, y_{2n_2}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pn_p})'$$

$$\beta = (\beta_1, \beta_2, \ldots, \beta_p)'$$

$$\varepsilon = (\varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{1n_1}, \varepsilon_{21}, \ldots, \varepsilon_{2n_2}, \ldots, \varepsilon_{p1}, \varepsilon_{p2}, \ldots, \varepsilon_{pn_p})'$$

One-way classification with fixed effect linear models of full rank:

where

$$X = \begin{pmatrix} \left.\begin{matrix} 1\ 0 \ldots 0 \\ \vdots\ \ \ddots\ \vdots \\ 1\ 0\ \ \ 0 \end{matrix}\right\} n_1 \text{ values} \\ \left.\begin{matrix} 0\ 1 \ldots 0 \\ \vdots\ \ \ddots\ \vdots \\ 0\ 1 \ldots 0 \end{matrix}\right\} n_2 \text{ values} \\ \vdots\ \ \vdots\ \ \ \vdots \\ \left.\begin{matrix} 0\ 0 \ldots 1 \\ \vdots\ \ \ddots\ \vdots \\ 0\ 0 \ldots 1 \end{matrix}\right\} n_p \text{ values} \end{pmatrix}$$

$$x_{ij} = \begin{cases} 1 & \textbf{if } \beta_i \textbf{ occurs in the } j^{th} \textbf{ observation } x_j \\ & \textbf{or if effect } \beta_i \textbf{ is present in } x_j \\ 0 & \textbf{if effect } \beta_i \textbf{ is absent in } x_j \end{cases}$$

$$n = \sum_{i=1}^{p} n_i .$$

One-way classification with fixed effect linear models of full rank:

So *X* is a matrix of order $n \times p,\ \beta$ **is fixed and**

- first *n_1* rows of ε **are** $\varepsilon_1^{'} = (1,0,0,...,0),$

- next *n_2* rows of ε **are** $\varepsilon_2^{'} = (0,1,0,...,0)$

- and similarly, the last *n_p* rows of ε **are** $\varepsilon_p^{'} = (0,0,...,0,1).$

Obviously, $rank\left(X\right) = p,\ E\left(Y\right) = X\beta$ **and** $Cov(Y) = \sigma^2 I.$

This completes the representation of a fixed effect linear model of full rank.

One-way classification with fixed effect linear models of full rank:

The null hypothesis of interest is $H_0 : \beta_1 = \beta_2 = ... = \beta_p = \beta$ **(say)**

and H_1 **: At least one** $\beta_i \neq \beta_j (i \neq j)$

where β **and** σ^2 **are unknown.**

We would develop here the likelihood ratio test.

It may be noted that the same test can also be derived through the least-squares method. This will be demonstrated later.

This way the readers will understand both the methods.

We already have developed the likelihood ratio for the hypothesis $H_0 : \beta_1 = \beta_2 = ... = \beta_p$ **in earlier case.**