# Relation between velocity and force manipulability ellipsoids of a rigid link manipulator

Shyam Sunder Nishad

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We have the following relations from the manipulator kinematics and dynamics theory:

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\boldsymbol{\theta}} \tag{1}$$

$$\boldsymbol{\tau} = \mathbf{J}^{\mathrm{T}}\mathbf{F} \tag{2}$$

where, $\mathbf{J}$ is the Jacobian relating the joint velocities $\dot{\boldsymbol{\theta}}$ with the end-effector velocities $\dot{\mathbf{x}}$. The joint torques are in the vector $\boldsymbol{\tau}$ and the force load at the end-effector is $\mathbf{F}$. From Eq. (2), we can write:

$$\boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\tau} = \mathbf{F}^{\mathrm{T}}\mathbf{J}\mathbf{J}^{\mathrm{T}}\mathbf{F} \tag{3}$$

For all the joint torques to be in a unit hyper-sphere, we have $\boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\tau} = 1$. Therefore,

$$\mathbf{F}^{\mathrm{T}}(\mathbf{J}\mathbf{J}^{\mathrm{T}})\mathbf{F} = 1 \tag{4}$$

Here, $\mathbf{J}\mathbf{J}^{\mathrm{T}}$ is a square, real symmetric matrix, and therefore can be diagonalized as:

$$\mathbf{J}\mathbf{J}^{\mathrm{T}} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{\mathrm{T}} \tag{5}$$

where, the diagonal matrix $\boldsymbol{\Lambda}$ contains the eigen values on the diagonals, and the matrix $\mathbf{S}$ contains the eigen vectors on its columns. Substituting this back in Eq. (4), we obtain:

$$(\mathbf{S}^{\mathrm{T}}\mathbf{F})^{\mathrm{T}}\boldsymbol{\Lambda}(\mathbf{S}^{\mathrm{T}}\mathbf{F}) = 1 \tag{6}$$

Let $\mathbf{S}^{\mathrm{T}}\mathbf{F} = \mathbf{f}$, which gives $\mathbf{F} = \mathbf{S}\mathbf{f}$. This implies, that coordinates $\mathbf{f}$ of the force vector $\mathbf{F}$ are written in the reference frame formed from the column vectors of $\mathbf{S}$ i.e. the eigen vectors of the matrix $\mathbf{J}\mathbf{J}^{\mathrm{T}}$. The substitution $\mathbf{S}^{\mathrm{T}}\mathbf{F} = \mathbf{f}$ in Eq. (6) gives:

$$\mathbf{f}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{f} = 1 \tag{7}$$

which can also be written in the form:

$$\lambda_1 f_1^2 + \lambda_2 f_2^2 + .... + \lambda_n f_n^2 = 1 \tag{8}$$

Thus, in the coordinate system formed by the eigen vectors $\mathbf{s}_1$, $\mathbf{s}_2$ ..., $\mathbf{s}_n$ (the column vectors of the matrix $\mathbf{S}$), the end-effector forces appear on the surface of an ellipsoid with axes lengths as $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$, ..., $\frac{1}{\lambda_n}$, if the joint torques are in a unit hyper-sphere. Here, $n$ is the dimension of the cartesian space (end-effector space).

Now, let's consider the velocity ellipsoid. Rewriting Eq. (1) as:

$$\dot{\boldsymbol{\theta}} = \mathbf{J}^{\#}\dot{\mathbf{x}} \tag{9}$$

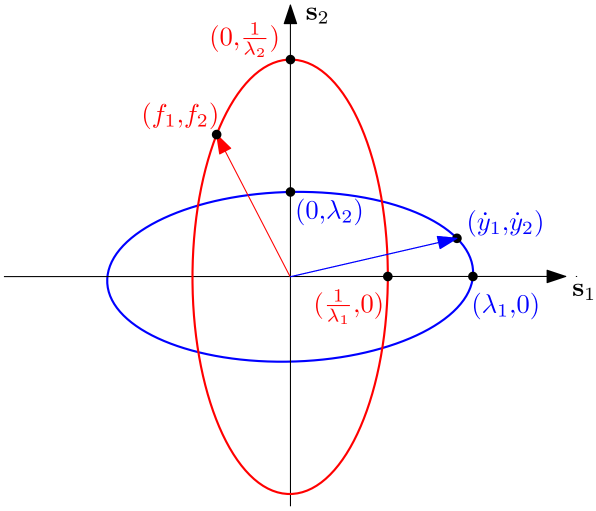


Figure 1: Force ellipsoid in red, velocity ellipsoid in blue for a two link-manipulator

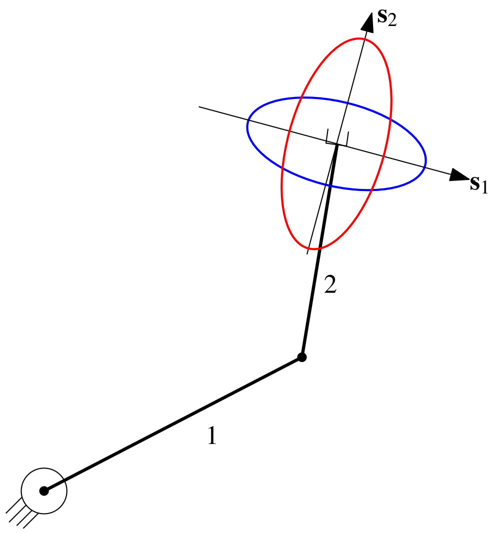


Figure 2: Force (red), velocity (blue) ellipsoids relative to a two link-manipulator: schematic

where, $\mathbf{J}^{\#}$ is the pseudo-inverse of $\mathbf{J}$. Thus,

$$\dot{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\boldsymbol{\theta}} = \dot{\mathbf{x}}\mathbf{J}^{\#^{\mathrm{T}}}\mathbf{J}^{\#}\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{J}\mathbf{J}^{\mathrm{T}})^{\#}\dot{\mathbf{x}} \tag{10}$$

For the joint velocities in a unit hyper-sphere, $\dot{\mathbf{x}}^{\mathrm{T}}\dot{\mathbf{x}} = 1$. Hence,

$$\dot{\mathbf{x}}(\mathbf{J}\mathbf{J}^{\mathrm{T}})^{\#}\dot{\mathbf{x}} = 1 \tag{11}$$

This equation is similar to Eq. (4), with $(\mathbf{J}\mathbf{J}^{\mathrm{T}})^{\#}$ appearing, which is inverse of $\mathbf{J}\mathbf{J}^{\mathrm{T}}$ in Eq. (4). Therefore, the eigen vectors of both the matrices are identical, while their eigen values are inverse of each other. Hence, proceeding as done previously for the force ellipsoid, we obtain:

$$\dot{\mathbf{y}}^{\mathrm{T}}\mathbf{\Lambda}^{-1}\dot{\mathbf{y}} = 1 \tag{12}$$

$$\text{where,}\, \dot{\mathbf{x}} = \mathbf{S}\dot{\mathbf{y}} \tag{13}$$

$$\frac{\dot{y}_1^2}{\lambda_1} + \frac{\dot{y}_2^2}{\lambda_2} + .... + \frac{\dot{y}_n^2}{\lambda_n} = 1 \tag{14}$$

Comparing the Eq. (14) with Eq. (8), we can say that the lengths of all the axes of the two ellipsoids are inversed. Specifically, in case of a two-link manipulator, if we say $\lambda_1 > \lambda_2$, we have:

$$\frac{\dot{y}_1^2}{\lambda_1} + \frac{\dot{y}_2^2}{\lambda_2} = 1 \tag{15}$$

$$\frac{f_1^2}{1/\lambda_1} + \frac{f_2^2}{1/\lambda_2} = 1 \tag{16}$$

Due to inversion of the lengths of the axes, the two ellipsoids appear orthogonal to each other (Fig. 1). Figure 2 shows the schematic of ellipsoids relative to the manipulator.