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SYSTEM IDENTIFICATION FROM NONUNIFORMLY SPACED SIGNAL MEASUREMENTS*

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Abstract. A couple of techniques for parameter estimation of a signal consisting of complex exponentials are presented. The system model employs the higher-order derivatives, or zero-initial conditioned integrals of the signal, together with the signal values. When the signal is sampled at nonuniformly distributed points, the orthogonal polynomial approximation and minimum error-variance criterion are used to compute all the values needed in the system models. The developed system models are demonstrated to give results better in accuracy than what can be obtained by employing Prony's method in a specific problem.

Zusammenfassung. Einige neue Techniken werden vorgestellt, mit denen die Parameter eines Signals geschätzt werden können, das aus komplexen Exponentiellen zusammengesetzt ist. Das Systemmodell verwendet höhere Ableitungen oder auch Integrale (mit der Anfangsbedingung Null) des Signals sowie das Signal selbst. Bei einer nicht-äquidistanten Signalabtastung werden eine Orthogonalpolynom-Approximation und das Kriterium kleinster Fehlervarianz verwendet, um sämtliche Werte zu berechnen, die in Systemmodell benötigt werden. Die so entwickelten Systemmodelle liefern, wie sich zeigen wird, genauere Ergebnisse als die, welche man mit dem Prony-Verfahren in einem spezifischer Problem erhalten kann.

Résumé. Un certain nombre de techniques pour l'estimation de paramètres d'un signal composé d'exponentielles complexes est présenté. Le modèle du système utilise les dérivées d'ordre supérieur ou les intégrales, initialement mises à zéro, du signal ainsi que le signal lui-même. Quand le signal est échantillonné aux points non uniformement distribués, l'approximation polynomiale orthogonale et le critère de minimisation de la variance de l'erreur sont utilisés pour calculer toutes les valeurs nécessaires pour modeliser le système. On démontre que les modèles de système développés connent des résultats meilleurs en précision que ceux que l'on peut obtenir par la méthode de Prony dans un problème spécifique.

Keywords. Parameter estimation, polynomial approximation, error-variance.

1. Introduction

Among various system identification problems, the characterization of the impulse response of a linear system by a sum of weighted complex exponentials, and then identifying/approximating the complex amplitudes and natural frequencies with high degree of accuracy has its special importance in a wide variety of applications. The study of the transient behavior of a system is one of them.

Whether it is the extraction of the poles and residues of a linear system which is excited by a unit impulse under zero initial conditions, from a finite set of output measurements; or it is the identification of the parameters of the impulse response from a measured finite length of input-output record of the system, by utilizing the Wiener-Hopf equation; the accurate determination of the complex natural frequencies of a signal consisting of complex exponentials is necessary.

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This paper deals with the case where the signal is sampled at nonuniformly distributed abscissae. Nonuniform sampling may be chosen over uniform sampling because of various reasons. For a 'burst'-type signal, the instantaneous measurement may vary rapidly over certain regions, and rather slowly outside. In this case, an efficient sampling scheme should sample the values of the signal densely only over the regions of rapid variation. Sampling the signal at the same rate when the rate of change of the signal is small does not seem very reasonable. Yet another reason in preferring nonuniform sampling to uniform sampling is the consideration of noise. Although it may not be possible to have control over the noise superimposed on the signal measurements, the signal-to-noise ratio (SNR) of the sampled set of values can be kept high by sampling more points of the signal far away from zero crossing. Here we assume that the expected value of noise is zero. Obviously, any sampling scheme to implement that will sample the signal at a set of nonequispaced points of the independent variable axis [1].

2. Parameter identification

A complex signal consisting of L exponentially damped sinusoids of arbitrary amplitudes, damping factors, and circular frequencies can be expressed as

$$f(t) = \sum_{i=1}^{L} A_i \exp\{s_i t\}, \tag{1}$$

where $s_i = \alpha_i + j\omega_i$ and the α_i's are negative numbers.

A real signal of the form of either 'sin' or 'cos',

$$f(t) = \sum_{i=1}^{M} B_i \exp\{\alpha_i t\} \begin{cases} \sin(\omega_i t + \theta_i) \\ \cos(\omega_i t + \theta_i), \end{cases} \tag{2}$$

can be transformed into equation (1), where now the $L = 2M$ values for the A_i or s_i will occur in complex conjugate pairs.

The parameter identification problem can be solved in two steps. First, the $2M$ roots for s_i are determined. Here we assume that the exact order of the system, $L = 2M$, is known a priori. Then, with the known values of s_i substituted, the $2M$ roots for A_i are found by solving a set of linear equations.

Once the poles s_i have been obtained, the amplitudes at the poles can be obtained by solving a linear least squares problem. Since the main objective of our paper is an accurate estimation of the poles from nonuniformly sampled data, we do not adequately address the problem of finding the A_i. This is because if the poles can be obtained with reasonable accuracy, then the amplitudes A_i can be solved from a linear least squares problem, with reasonable accuracy.

3. Determination of complex natural frequencies

Several system models to determine the values of s_i are described below. The first model employs the equispaced sampled values of the signal [3, 4]. This model is well known, and was originally presented by Prony [4]. The other models utilize the values of the high-order derivatives, or zero-initial conditioned integrals of the signal, together with the signal values. These models are being introduced here for the purpose of identification, and we like to investigate how they perform comparatively in real situations.

System Model I

Suppose that the signal values are given at $4M$ equispaced points of the independent variable axis.

Substituting $t = k\Delta t$, $k = 0, 1, \ldots, 4M-1$, in equation (1), where Δt is the sampling interval, and then with a change of variable in the following way:

$$z_i = e^{s_i \Delta t}, \tag{3}$$

the $4M$ discrete values of the signal will be given as

$$f_k = \sum_{i=1}^{2M} A_i z_i^k, \quad k = 0, 1, \ldots, 4M-1. \tag{4}$$

Let us assume that the $2M$ solutions for z_i are the roots of the polynomial equation

$$z^{2M} + a_{2M-1} z^{2M-1} + \ldots + a_1 z + a_0 = 0 \tag{5}$$

with $a_{2M} = 1$.

Thus, the solutions for z_i can be obtained provided the values for a_i are known.

By utilizing equations (4) and (5), a set of linear equations can be written in matrix form as shown below:

$$\left[\begin{array}{cccc|c} f_0 & f_1 & \cdots & f_{2M-1} & f_{2M} \\ f_1 & f_2 & \cdots & f_{2M} & f_{2M+1} \\ \vdots & \vdots & & \vdots & \vdots \\ f_{2M-1} & f_{2M} & \cdots & f_{4M-2} & f_{4M-1} \end{array}\right] \left[\begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_{2M-1} \\ \hline a_{2M} \end{array}\right] = \mathbf{0}. \tag{6}$$

The only solution of equation (6) is the trivial solution, unless one of the coefficients is set to unity. With $a_{2M} = 1$, equation (6) can be rewritten as

$$\begin{bmatrix} f_0 & f_1 & \cdots & f_{2M-1} \\ f_1 & f_2 & \cdots & f_{2M} \\ \vdots & \vdots & & \vdots \\ f_{2M-1} & f_{2M} & \cdots & f_{4M-2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2M-1} \end{bmatrix} = - \begin{bmatrix} f_{2M} \\ f_{2M+1} \\ \vdots \\ f_{4M-1} \end{bmatrix}, \tag{7}$$

which can be solved for a_i by the conjugate gradient method [7, 10]. The conjugate gradient method has the advantage of a direct method and that of an iterative method. It has the advantage of a direct method as the solution can be obtained in a finite number of steps starting with any initial guess. Also, it has the advantage of an iterative method as the round-off and truncation errors do not propagate and hence any 'ill-conditioning' of the system matrix only reduces the rate of convergence and not the accuracy in the solution. This is because the conjugate gradient method can also be applied to exactly singular systems [2]. In that case, we obtain the minimum norm solution.

From equation (3), the solutions for s_i will be given as follows:

$$s_i = [\tfrac{1}{2} \ln(|z_i|^2) \pm j \arg(z_i)]/\Delta t,$$

where $|\cdot|$ and $\arg(\cdot)$ denote the magnitude and argument of the complex variable respectively.

System Model II

Now assume that the signal value and all its derivatives up to the order $4M-1$, evaluated at a single point t_0 of the t-axis, are available.

Then, utilizing equation (1) and performing differentiation, the required number of times in each case, the signal and all its derivatives at $t = t_0$ will be given as follows:

$$f^n = \left.\frac{\mathrm{d}^n[f(t)]}{\mathrm{d}t^n}\right|_{t=t_0} = \sum_{i=1}^{2M} s_i^n A_i \exp\{s_i t_0\}, \quad n = 0, 1, \ldots, 4M-1. \tag{8}$$

Now we will use the same technique that we have used before. Let us assume that the $2M$ solutions for s_i are the roots of the polynomial equation

$$s^{2M} + a_{2M-1}\, s^{2M-1} + \ldots + a_1 s + a_0 = 0, \tag{9}$$

with $a_{2M} = 1$.

Then, utilizing equations (8) and (9), a matrix equation similar to equation (6) can be written as

$$\left[\begin{array}{cccc|c} f^0 & f^1 & \ldots & f^{2M-1} & f^{2M} \\ f^1 & f^2 & \ldots & f^{2M} & f^{2M+1} \\ \vdots & \vdots & & \vdots & \vdots \\ f^{2M-1} & f^{2M} & \ldots & f^{4M-2} & f^{4M-1} \end{array}\right] \left[\begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_{2M-1} \\ \hline a_{2M} \end{array}\right] = \mathbf{0}, \tag{10}$$

or as

$$\begin{bmatrix} f^0 & f^1 & \ldots & f^{2M-1} \\ f^1 & f^2 & \ldots & f^{2M} \\ \vdots & \vdots & & \vdots \\ f^{2M-1} & f^{2M} & \ldots & f^{4M-2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2M-1} \end{bmatrix} = - \begin{bmatrix} f^{2M} \\ f^{2M+1} \\ \vdots \\ f^{4M-1} \end{bmatrix}, \tag{11}$$

with $a_{2M} = 1$.

Equation (11) can be solved for a_i by the conjugate gradient method, and then, from equation (9), the values for s_i can be found by solving a polynomial equation.

System Model III

As a variation of the previous case, let us assume that the signal values and all its derivatives up to order 2M are known at arbitrarily chosen $2M$ distinct points, $t_0, t_1, \ldots, t_{2M-1}$ of the independent variable axis.

In this case, equation (10) can be rewritten as

$$\left[\begin{array}{cccc|c} f_0 & f_0^1 & \ldots & f_0^{2M-1} & f_0^{2M} \\ f_1 & f_1^1 & \ldots & f_1^{2M-1} & f_1^{2M} \\ \vdots & \vdots & & \vdots & \vdots \\ f_{2M-1} & f_{2M-1}^1 & \ldots & f_{2M-1}^{2M-1} & f_{2M-1}^{2M} \end{array}\right] \left[\begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_{2M-1} \\ \hline a_{2M} \end{array}\right] = \mathbf{0}, \tag{12}$$

where f_k^n is the nth-order derivative of the signal evaluated at $t = t_k$.

With $a_{2M} = 1$, equation (12) becomes

$$\begin{bmatrix} f_0 & f_0^1 & \ldots & f_0^{2M-1} \\ f_1 & f_1^1 & \ldots & f_1^{2M-1} \\ \vdots & \vdots & & \vdots \\ f_{2M-1} & f_{2M-1}^1 & \ldots & f_{2M-1}^{2M-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2M-1} \end{bmatrix} = - \begin{bmatrix} f_0^{2M} \\ f_1^{2M} \\ \vdots \\ f_{2M-1}^{2M} \end{bmatrix}, \tag{13}$$

which is solved for a_i by the conjugate gradient method.

System Model IV

Integration is the inverse process of differentiation. So it will be possible to solve for a_i when the signal values and their high-order indefinite integrals, instead of derivatives, are known at several points of the independent variable axis.

But, first, we have to find a way to relate the indefinite integral to the definite integral of a function. Namely, it is the definite integral of a function that can be interpreted as the area under the function curve, and accordingly evaluated when the function values are known.

Let us define the antiderivatives $f^{-1}(t), f^{-2}(t), \ldots, f^{-2M}(t)$ of the signal $f(t)$ in the following way:

$$\frac{\mathrm{d}f^{-(n+1)}(t)}{\mathrm{d}t} = f^{-n}(t); \quad n = 0, 1, \ldots, 2M-1. \tag{14}$$

Then, similar to each row of the matrix equation (12), we can write

$$a_{2M} f(t) + a_{2M-1} f^{-1}(t) + \cdots + a_1 f^{-(2M-1)}(t) + a_0 f^{-2M}(t) = 0 \quad \text{for} \quad \text{any } t, \tag{15}$$

where the $2M$ solutions for s_i now are the roots of the polynomial equation

$$a_{2M} s^{2M} + a_{2M-1} s^{2M-1} + \cdots + a_1 s + a_0 = 0.$$

Integrating both sides of the recurrence relation (14) for the successive values of n over the interval $[t_0, t]$, where t_0 is the initial time, we get the following set of equations:

$$f^{-1}(t) = f^{-1}(t_0) + \int_{t_0}^{t} f(t)\,\mathrm{d}t,$$

$$f^{-2}(t) = f^{-2}(t_0) + (t-t_0) f^{-1}(t_0) + \iint_{t_0}^{t} f(t)\,\mathrm{d}t^2,$$

$$f^{-3}(t) = f^{-3}(t_0) + (t-t_0) f^{-2}(t_0) + \frac{(t-t_0)^2}{2!} f^{-1}(t_0) + \iiint_{t_0}^{t} f(t)\,\mathrm{d}t^3,$$

$$\vdots$$

$$f^{-2M}(t) = f^{-2M}(t_0) + (t-t_0) f^{-(2M-1)}(t_0) + \frac{(t-t_0)^2}{2!} f^{-(2M-2)}(t_0)$$

$$+ \cdots + \frac{(t-t_0)^{2M-1}}{(2M-1)!} f^{-1}(t_0) + \underbrace{\int_{t_0}^{t} \cdots \int}_{2M \text{ times}} f(t)\,\mathrm{d}t^{2M}, \tag{16}$$

where the initial values $f^{-1}(t_0), f^{-2}(t_0), \ldots, f^{-2M}(t_0)$ are not known.

Let us assume that all zero-initial conditioned definite integrals of the signal, up to order $2M$, are available at $4M$ distinct points $t_1, t_2, \ldots, t_{4M}$ of the t-axis.

Then, utilizing equations (15) and (16), and using the notation

$$f_0^{-n}(t_k) = \underbrace{\int_{t_0}^{t_k} \cdots \int}_{n \text{ times}} f(t)\,\mathrm{d}t^n, \quad n = 1, 2, \ldots, 2M, \qquad k = 1, 2, \ldots, 4M, \tag{17}$$

where the suffix 0 is used to emphasize that these are the zero-initial conditioned high-order integrals of the signal, a matrix equation can be written in the following way:

$$\left[\begin{array}{c|c|c} f(t_1) & & \\ f(t_2) & C & D \\ \vdots & & \\ f(t_{4M}) & & \end{array}\right]\left[\begin{array}{c} a_{2M} \\ \hline a_{2M-1} \\ \vdots \\ a_0 \\ \hline \boldsymbol{b} \end{array}\right]=\mathbf{0}, \tag{18}$$

where

$$C=\begin{bmatrix} f_0^{-1}(t_1) & f_0^{-2}(t_1) & \dots & f_0^{-2M}(t_1) \\ f_0^{-1}(t_2) & f_0^{-2}(t_2) & \dots & f_0^{-2M}(t_2) \\ \vdots & \vdots & & \vdots \\ f_0^{-1}(t_{4M}) & f_0^{-2}(t_{4M}) & \dots & f_0^{-2M}(t_{4M}) \end{bmatrix},$$

$$D=\begin{bmatrix} 1 & \dfrac{(t_1-t_0)}{1!} & \dfrac{(t_1-t_0)^2}{2!} & \dots & \dfrac{(t_1-t_0)^{2M-1}}{(2M-1)!} \\ 1 & \dfrac{(t_2-t_0)}{1!} & \dfrac{(t_2-t_0)^2}{2!} & \dots & \dfrac{(t_2-t_0)^{2M-1}}{(2M-1)!} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \dfrac{(t_{4M}-t_0)}{1!} & \dfrac{(t_{4M}-t_0)^2}{2!} & \dots & \dfrac{(t_{4M}-t_0)^{2M-1}}{(2M-1)!} \end{bmatrix},$$

and

$$\boldsymbol{b}=\begin{bmatrix} a_{2M-1}f^{-1}(t_0)+a_{2M-2}f^{-2}(t_0)+\cdots+a_0f^{-2M}(t_0) \\ a_{2M-2}f^{-1}(t_0)+a_{2M-3}f^{-2}(t_0)+\cdots+a_0f^{-(2M-1)}(t_0) \\ \vdots \\ a_1f^{-1}(t_0)+a_0f^{-2}(t_0) \\ a_0f^{-1}(t_0) \end{bmatrix}.$$

With $a_{2M}=1$, equation (18) can be rewritten as follows:

$$[C \quad D]\left[\begin{array}{c} a_{2M-1} \\ \vdots \\ a_0 \\ \hline \boldsymbol{b} \end{array}\right]=-\begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_{4M}) \end{bmatrix}, \tag{19}$$

which is solved for $a_{2M-1},\dots,a_0$ by the conjugate gradient method. Note that we are not interested in the $\boldsymbol{b}$ component of the solution vector, which contains the $2M$ initial conditions of the integrals.

4. Comparison of different system models

When exact data values are fed in the system models discussed, they should all produce identical results under ideal conditions. The conjugate gradient method for solving the system equations is utilized. The algorithms are guaranteed to converge in all possible cases. However, the reliability of the solution may be questionable when the system matrix is highly ill-conditioned and the data used are inaccurate [6]. Inaccuracies in the data may be introduced in the form of error of approximation when only nonuniformly sampled values of the signal are available, or/and in the form of measurement error when random noise is superimposed on the signal.

The system matrices involved in all system models, except that in Model II, are naturally ill-conditioned. This is because the values of the elements in the successive rows of the matrices vary only slightly, which makes the difference terms of the unknown variables weakly represented [3]. To avoid this problem, however, we may write the system equations in overdetermined form, and then find a least-squares solution in the following way:

$$\textbf{if}\;\; V_{m\times l}\boldsymbol{a}_{l\times 1}=\boldsymbol{u}_{m\times 1},\;\; m>l\;\; \textbf{then}\;\; \boldsymbol{a}=(V^{\mathrm{T}}V)^{-1}V^{\mathrm{T}}\boldsymbol{u}. \tag{20}$$

When utilizing the conjugate gradient method as discussed above, we do not explicitly form the normal equations, even though in actuality we are solving the normal least-squares problem.

In Prony's model, the ill-conditioning of the system matrix also arises due to the closeness of values of the elements in the successive columns, which makes the sum terms of the unknown variables weakly represented. Nothing can be done to compensate the ill-conditioning when the sampled values are given. This is why we utilize the conjugate gradient method to solve the matrix equations.

Requirement of all derivatives, up to order $4M-1$, for a system of order $2M$, does not make Model II very suitable in practice. Higher dimensionality of the matrix in Model IV can increase its ill-conditioning. A constant factor may have to be taken out of s_i-roots by dividing and multiplying the successive columns of the matrices in Models III and IV respectively, with the powers of the constant. In Model IV, we introduce different scale factors in the columns of matrix D—equation (19). Remember that, in this case, we are not interested in the last $2M$ components of the solution vector. It is important to point out that the matrix equations encountered in all the techniques of interest are highly ill-conditioned. Here, the measure of ill-conditioning is determined by the condition number of the matrix. The condition number is defined as the ratio of the moduli of the largest and smallest eigenvalues. Even though the condition number can be reduced by suitable choice of the sampling interval, but for a given data set, one is stuck with the condition number. The only thing that can be done is to choose a solution procedure that is robust to round-off and truncation errors. In our solution procedure, we utilize scaling and over-determination as discussed above, together with the conjugate gradient method.

5. Nonuniform sampling and orthogonal polynomial approximation

By utilizing the concept of function space and the related idea of finding a set of suitable basis to span a given function space, a continuous signal $f(t)$ considered over a finite range of the independent variable axis can be approximated as follows:

$$f(t)=\sum_{j=1}^{N} c_j p_{j-1}(t), \tag{21}$$

where $p_{j-1}(t)$ is a polynomial of degree $j-1$ in t, and c_j is its coefficient.

A suitable polynomial approximation, in this case, is guaranteed by the classical theorem of Weierstrass; and for the complete reconstruction of the signal, it is necessary that the sampling theorem be satisfied locally [1]. Particularly convenient is the case where the generating polynomials constitute an orthogonal set over the collection of sampled points.

Ralston [5] developed a series of polynomials which satisfy a generalized orthogonality condition over a set of arbitrary distributed abscissas. The polynomials can be evaluated from the recurrence relation

$$p_j = (t - a_j)p_{j-1} - b_{j-1}p_{j-2}, \quad j \geq 1 \tag{22}$$

with $p_0 = 1$, $p_{-1} = 0$,

and

$$a_j = \frac{1}{D_{j-1}} \sum_{k=1}^{K} t_k [p_{j-1}(t_k)]^2,$$

$$b_j = \frac{D_j}{D_{j-1}},$$

$$D_j = \sum_{k=1}^{K} [p_j(t_k)]^2,$$

where $\{t_k\colon k = 1,2,\ldots, K\}$ are not necessarily at uniform spacings.

The generating polynomials for the successive-order derivatives of the signal are given by the following relations:

$$p_j^1 = p_{j-1} + (t - a_j)p_{j-1}^1 - b_{j-1}p_{j-2}^1, \quad j \geq 2,$$

with $p_0^1 = 0$, $p_1^1 = 1$, (23)

and

$$p_j^n = np_{j-1}^{n-1} + (t - a_j)p_{j-1}^n - b_{j-1}p_{j-2}^n, \quad n \geq 2,\ j \geq 2,$$

with $p_0^n = 0$, $p_1^n = 0$,

where n is the order of derivative, and a_j, b_j, and D_j are defined as in equation (22).

To accurately evaluate the zero-initial conditioned high-order integrals of the signal, we proceed one step at a time. Each sampling interval is divided into a large number of equispaced subintervals. The orthogonal polynomial approximation is used to compute the function values at all intermediate points, and then the composite trapezoidal rule is applied to evaluate the area enclosed between the successive sampled points.

Integrating the polynomial approximation analytically does not seem very convenient because we do not find any straightforward recurrence relation for the generating functions; whereas the procedure presented here can be easily mechanized on a computer, even for a large number of sample points.

In the presence of noise mixed with the sampled signal values, the series (21) should be truncated at the order of approximation where the error-variance

$$\sigma_J^2 = \frac{\sum_{k=1}^{K} \left[f(t_k) - \sum_{j=1}^{J} c_j p_{j-1}(t_k) \right]^2}{K - J}, \quad J \leq N \leq K, \tag{24}$$

is either minimum or does not decrease appreciably any further with the increase of approximation order. We assumed here that the noise added to the signal values is of zero mean, and uncorrelated. Then, it can be shown that the expected value of mean-square error is the ratio of the sum of square deviations and residual degrees of freedom [11].

The errors of approximation in the computed values of the signal comprising of two damped sinusoids in presence of noise are shown in Fig. 1. The error-variance plots are shown in Fig. 2. A vertical line is drawn at the order of approximation where the polynomial series is truncated.

Since the polynomials are computed by recurrence relation, the quantization error introduced in the higher-order polynomials can be a problem while processing too large number of data points and/or the additive noise is not purely white; it is then advisable to restrict the value of N, so that the region of oscillation in the error-variance plot, as shown in Fig. 2(b), can be avoided.

6. Simulation study

Example 1

The signal $f(t)=\exp\{-t\}\sin(2t)+\exp\{-1.5t\}\sin(5t)$ is sampled at 20 nonuniformly distributed abscissae. The choice of the sampled points is completely arbitrary except that no sampling interval exceeds $1/(8f)$, where f is the highest frequency present in the signal.

The orthogonal polynomial approximation and minimum error-variance criterion are used to find a closed-form expression for the signal.

The polynomial series is truncated at the order of approximation 13, where the error-variance of approximation gets the minimum value of 0.269×10^{-11}. The truncated polynomial series then provides all the values needed in various system models. The least-squares problems are solved by the conjugate gradient method. The computed values of the s_i-root are shown in Table 1. The exact values are shown for comparison.

Table 1

Models employed	The s_i-roots computed	Exact values
Prony's model	$-0.51\pm1.93j$	
	$-1.58\pm4.87j$	
Model III (derivatives)	$-0.99\pm1.98j$	$-1.0\pm2.0j$
	$-1.54\pm5.02j$	$-1.5\pm5.0j$
Model IV (integrals)	$-1.05\pm1.98j$	
	$-1.50\pm5.04j$	

To obtain the results by Prony's model, the signal values were computed at equispaced points, using the polynomial expression for the signal. The s_i-roots computed by Prony's model are found to be less accurate than those computed by the system models developed.

By a utility subroutine, we also measured the ratio of the largest to least eigenvalues of the normal matrices in each model. This ratio is a relative measure of the sensitivity of the least-squares solution with respect to errors in the data values fed, and the ratios are 2.49×10^6, 1.21×10^3, and 3.15×10^5 for Prony's, the derivative, and the integral models respectively.

Example 2

In this example, we sample the signal $f(t)=\exp\{-0.3t\}\sin(2t)+\exp\{-0.5t\}\sin(5t)$ at 60 arbitrarily distributed points of the independent variable axis. The zero mean uncorrelated computer synthesized

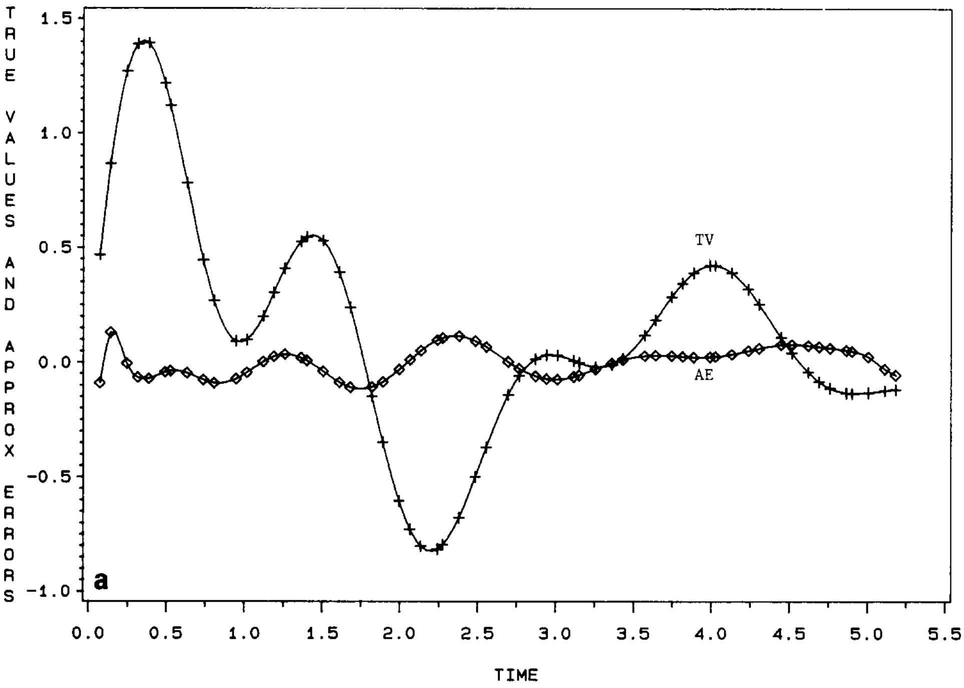

Fig. 1. (a) Orthogonal polynomial approximation; signal $= \exp\{-0.3t\}\sin(2t)+\exp\{-0.5t\}\sin(5t)$, SNR = 10 dB.

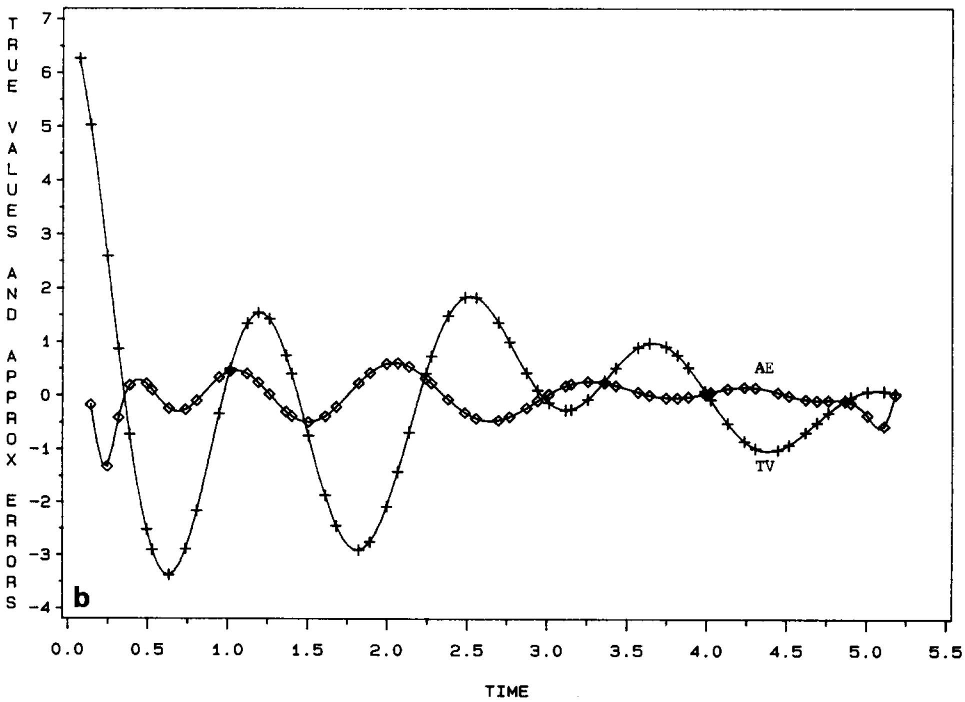

Fig. 1. (b) Orthogonal polynomial approximation; first-order derivatives.

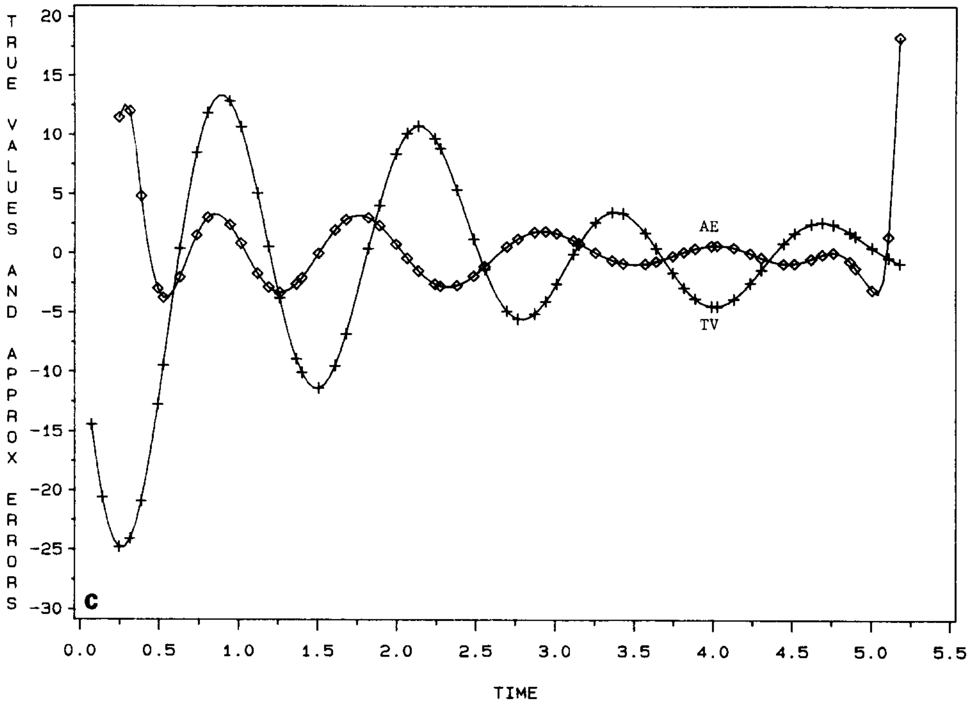

Fig. 1. (c) Orthogonal polynomial approximation; second-order derivatives.

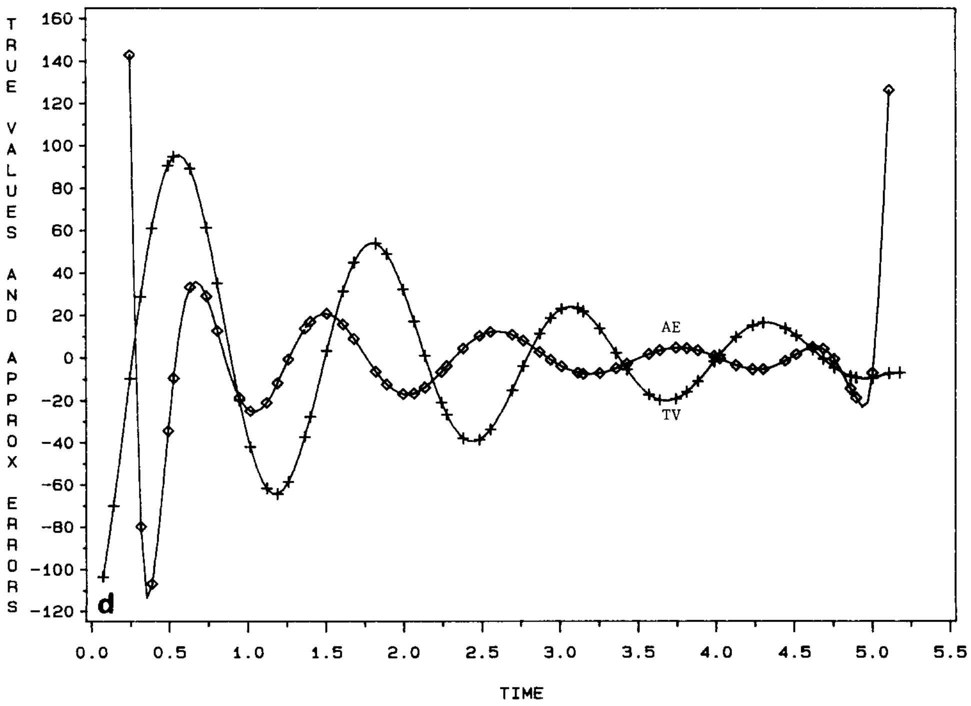

Fig. 1. (d) Orthogonal polynomial approximation; third-order derivatives.

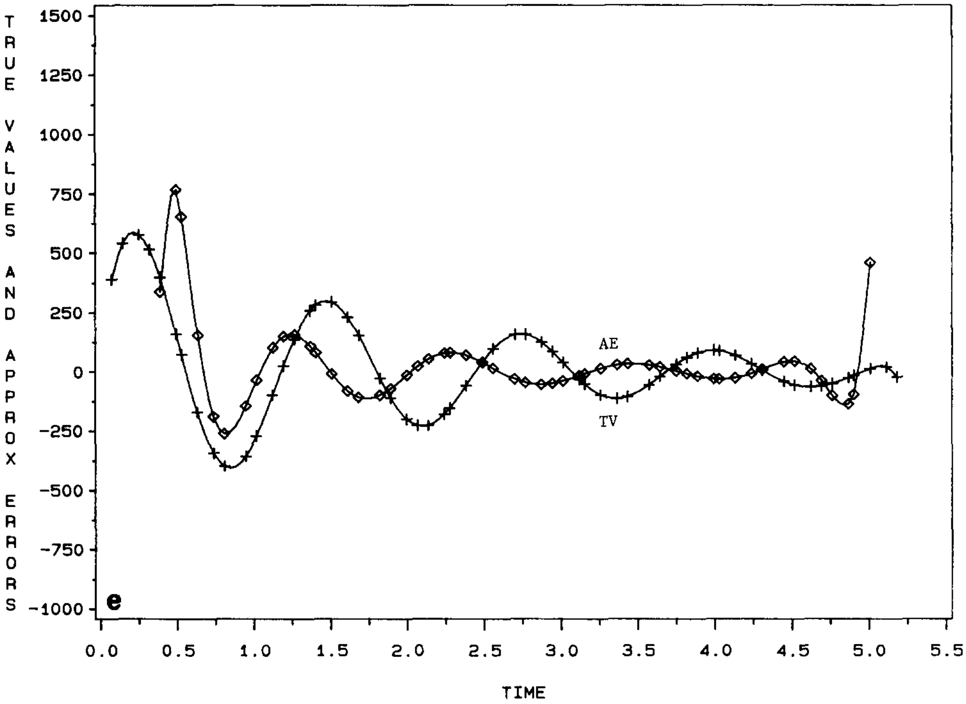

Fig. 1. (e) Orthogonal polynomial approximation; fourth-order derivatives.

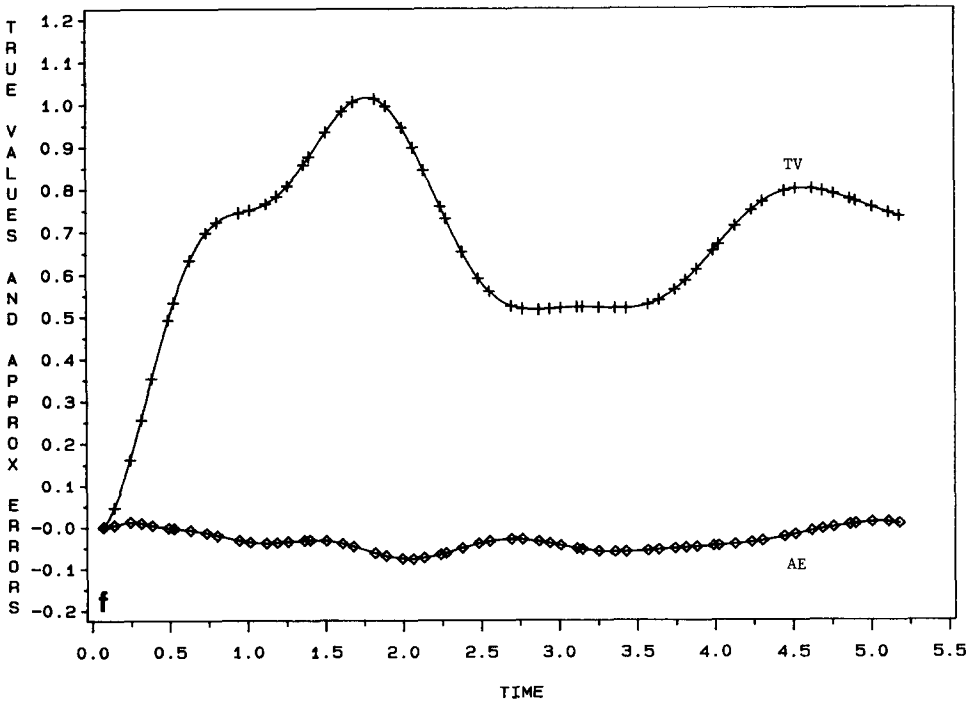

Fig. 1. (f) Orthogonal polynomial approximation; first-order integrals with zero I.C.

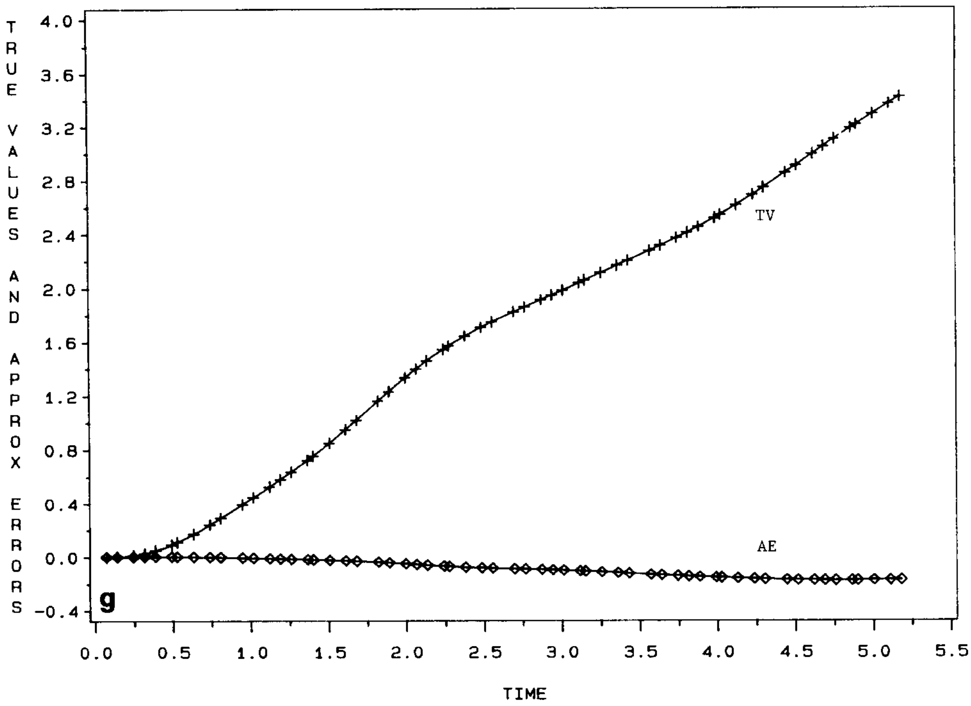

Fig. 1. (g) Orthogonal polynomial approximation; second-order integrals with zero I.C.

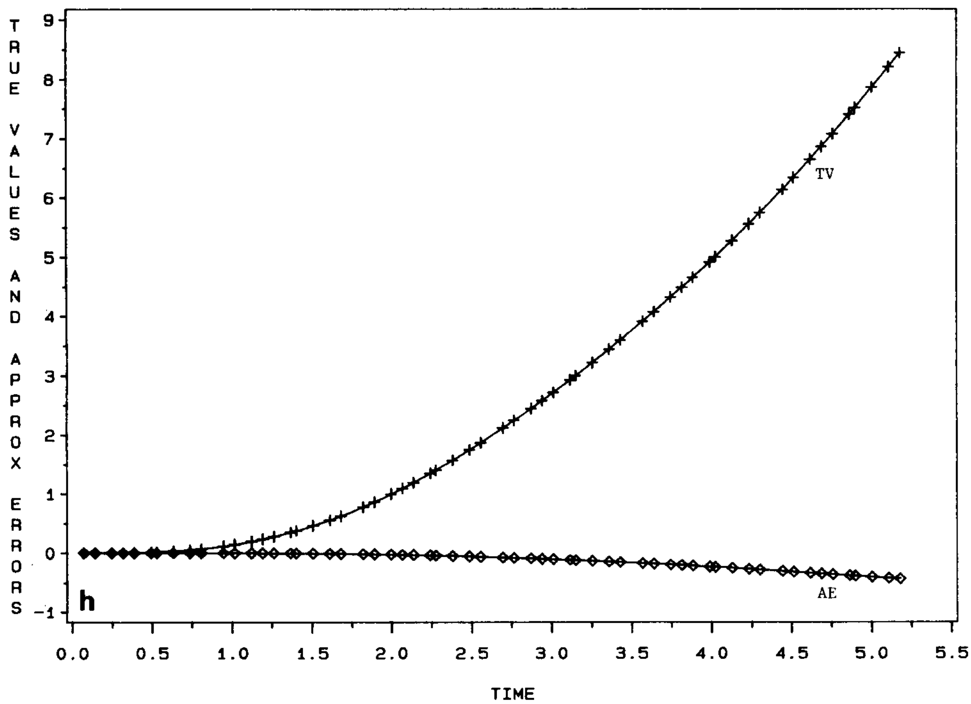

Fig. 1. (h) Orthogonal polynomial approximation; third-order integrals with zero I.C.

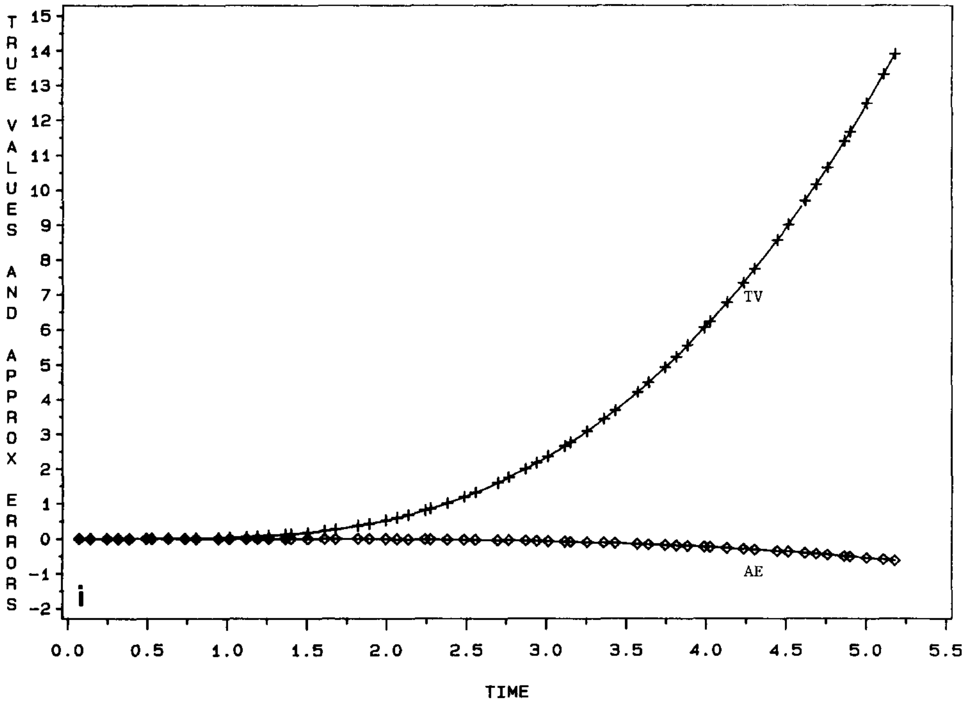

Fig. 1. (i) Orthogonal polynomial approximation; fourth-order integrals with zero I.C.

noise is then mixed with the signal values, setting the signal-to-noise ratio at 40, 20, and and 10 dB levels. In each case, Models III and IV are employed to compute the s_i-values, as shown in Table 2.

The values of derivatives computed at the middle 40 points are fed in Model III, whereas Model IV uses the zero-initial conditioned integrals evaluated at the first 40 points. When noise is present, the errors of approximation in the derivatives may become large at the ends of sampling span, and those in the integral values may accumulate over the sampling span. The variations of approximation errors in the computed values, when the SNR level is set at 10 dB, is shown in Fig. 1. The error-variance versus approximation order plots are shown in Fig. 2. Please note that we have avoided the region of oscillation in the error-variance plot of Figure 2(b), which is caused by the quantization errors in the higher-order polynomials as explained above.

Table 2

Models employed	The s_i-values computed				
	No noise	SNR = 40 dB	SNR = 20 dB	SNR = 10 dB	Exact values
Model III (derivatives)	$-0.3001 \pm 1.9998j$ $-0.4999 \pm 4.9999j$	$-0.313 \pm 1.92j$ $-0.484 \pm 4.96j$	$-0.281 \pm 2.03j$ $-0.471 \pm 4.96j$	$-0.22 \pm 2.16j$ $-0.81 \pm 5.30j$	$-0.3 \pm 2.0j$
Model IV (integrals)	$-0.2993 \pm 1.9997j$ $-0.5009 \pm 4.9997j$	$-0.299 \pm 1.99j$ $-0.484 \pm 5.03j$	$-0.292 \pm 1.98j$ $-0.458 \pm 5.06j$	$-0.17 \pm 2.14j$ $-0.55 \pm 5.07j$	$-0.5 \pm 5.0j$

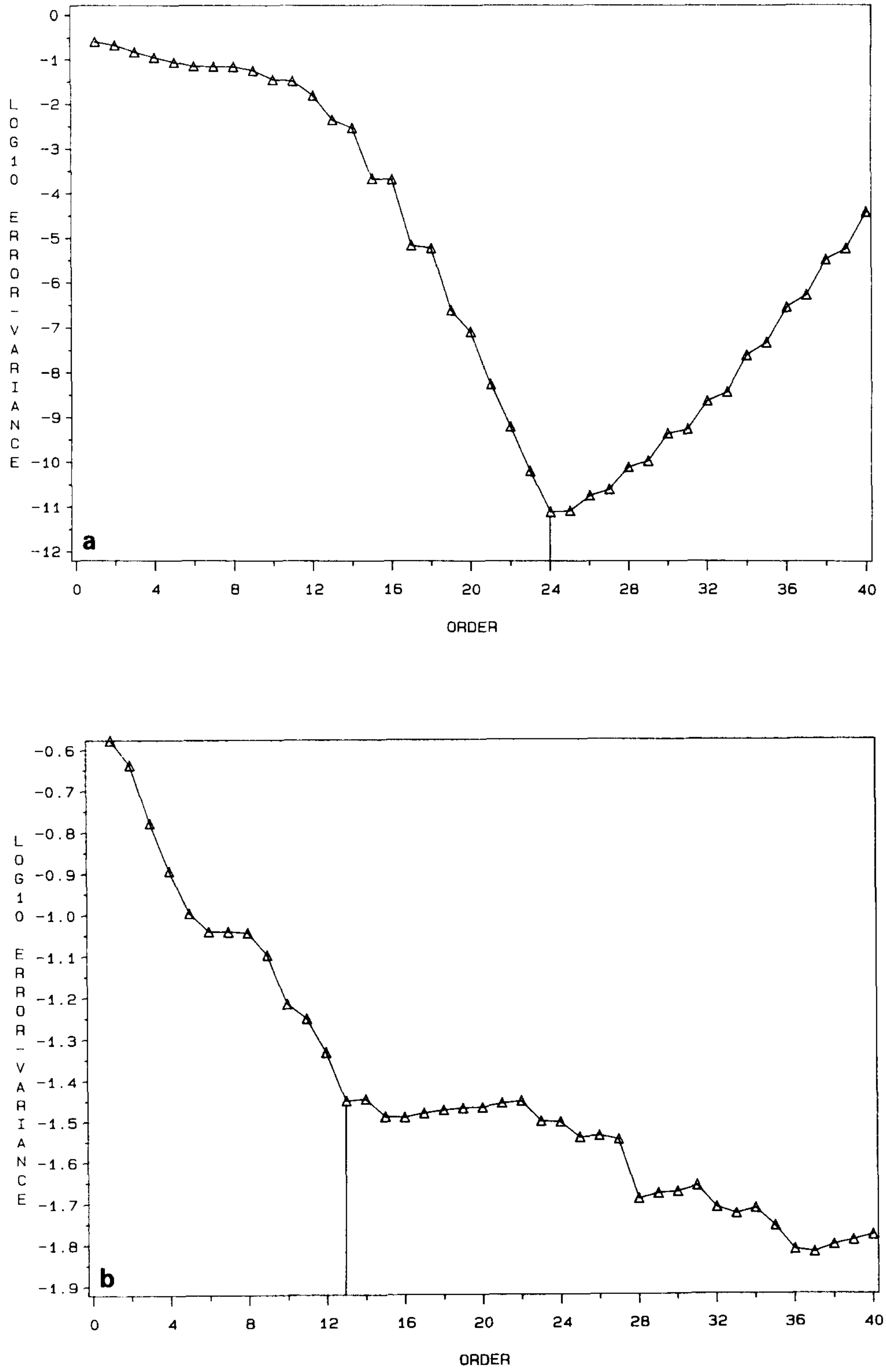

Fig. 2. Error-variance plot. (a) No noise. (b) SNR = 10 dB.

7. Concluding remarks

In this paper, we have demonstrated how the measurement interpolation using the orthogonal polynomial approximation and minimum error-variance criterion can be advantageously combined with the system identification problem and integrodifferential techniques can be utilized for parameter estimation purpose.

The use of measurement interpolation in the estimation problem brings many advantages. First of all, one does not have to sample a signal at uniform spacings then. This feature incorporates all the merits of nonuniform sampling. Second, the method can be effectively utilized to filter a signal corrupted with white noise, as demonstrated in [8], where the Monte Carlo comparisons of the derivative and integral methods with Prony's, Pisarenko's, and the Pencil-of-Function methods are also presented.

Numerical ill-conditioning associated with the system identification problem, which makes the solution of a system equation extremely sensitive to any error in data, is briefly mentioned in this paper. The interested reader is requested to kindly refer to [9], where a survey of ill-conditioning and its effects in the estimation problem is presented. It is worth mentioning here that an accurate solution from a highly ill-conditioned equation can only be expected when the data utilized is also very accurate.

Please note that for effective utilization of the measurement interpolation, as demonstrated in this paper, the signal must be sampled at a relatively large number of points; otherwise, we may not get an accurate polynomial approximation for the signal when it is corrupted with noise.

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