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Chaos Synchronization

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6.1 INTRODUCTION

Chaos is one of the most appealing topics for researchers from various subject backgrounds having non-linear dynamics as a common interest. This is due, primarily, to its significance as a tool which one can use to make sense of a complex world. Examples of chaos include weather patterns, biological systems, fluid turbulence, mobile communications, traffic, population dynamics, and astrophysics – to name just a few. Synchronized chaotic systems have received a lot of attention due to their potential use for secure communications. Synchronized semiconductor lasers are especially attractive due to their potential use in all-optical communication systems. In this book, chaos generation in semiconductor lasers with optical feedback is discussed in Chapter 2 as one of the regimes of operation of these systems. The use of chaotic semiconductor lasers in all-optical secure communication systems is discussed in Chapter 9.

Historically, synchronization in non-chaotic systems was realized before that of chaotic systems. Christiaan Huygens first observed anti-phase synchronization of two pendulum clocks, with a common frame, in 1665. This was the subject of some of the earliest deliberations of the Royal Society. Huygens found that the pendulum clocks swung at exactly the same frequency and 180° out of phase. When he disturbed one pendulum the anti-phase state was restored within half an hour and pendulum clocks remained synchronized indefinitely, thereafter, if left undisturbed. He found that synchronization did not occur when the clocks were separated beyond a certain distance, or oscillated in mutually perpendicular planes. Huygens deduced that the crucial interaction came from very small movements of the common frame supporting the two clocks. He also provided a physical explanation for how the frame motion set up the anti-phase motion [1].

The synchronization of chaotic systems is a subject with a long history that has been a ‘hot topic’ in the past ten years. Fujisaka and Yamada [2–5] did early work on synchronization of chaotic systems, but it was not until the work of Pecora and Carroll [6, 7] that the subject received a significant amount of attention. The term ‘chaotic synchronization’ refers to a
variety of phenomena in which chaotic systems adjust a given property of their motion to
a common behaviour due to a coupling or to a driving force [8]. The systems might be
identical or different, the coupling might be unidirectional (master–slave or drive–response
coupling) or bi-directional (mutual coupling) and the driving force might be deterministic or
stochastic.

Intrinsic interest in synchronization phenomena from a dynamical systems and chaos theory
point of view, and practical applications in secure communications [9], have spurred a wide
range of research studies. Two systems are coupled unidirectionally if the dynamics of one
system (called master or drive) affects the dynamics of the other (called slave or response),
while the dynamics of the slave does not affect the dynamics of the master. The fact that two
unidirectionally coupled chaotic systems can be used in a secure communication scheme was
first shown by Cuomo and Oppenheim [10, 11], who built a circuit version of the Lorenz
equations and showed the possibility of using this system to transmit a small speech signal.
While some years ago the word ‘chaos’ had a negative connotation in applied research, now
a days researchers are studying techniques for taking advantage of chaotic dynamics instead
of trying to avoid it. The use of chaos synchronization in secure communication systems is
a good example of this modern point of view.

Chaos synchronization includes a number of different phenomena such as (1) completely
identical oscillations in the coupled systems (complete synchronization); (2) frequency
locking (frequency synchronization); (3) phase locking between the two systems while their
amplitudes remain uncorrelated (phase synchronization); (4) the output of one system is
correlated with the output of the other system but it lags in time (lag synchronization); and
(5) the outputs of the two systems are functionally related (generalized synchronization).
The latter usually occurs in real experiments of chaos synchronization, in which no complete
synchronization is observed due to parameter mismatches. In each case, one can separate
cases of full and partial synchronization and, thus, several measures have been proposed
to determine the degree of synchronization. (these measures include correlation function,
mutual information, synchronization diagrams, etc.) A recent and complete review of chaotic
synchronization phenomena can be found in [8].

In the field of lasers, the first experimental demonstrations of synchronization of two
chaotic lasers were performed using Nd: YAG [12] and CO₂ lasers [13, 14]. In the scheme
used by Roy and Thornburg [12], one or both of the lasers were driven into chaos by periodic
modulation of their pump beams. Sugawara et al. [13] demonstrated synchronization of two
chaotic passive Q-switched lasers by modulating the saturable absorber in the cavity of one
laser with the output of the other laser. Liu et al. [14] used two passive, optically-coupled,
Q-switched lasers, and the amount of coupling used induced transitions from unsynchronized
periodic oscillations to synchronized chaotic ones. Colet and Roy [15] were the first to
propose a scheme for encoding data within a chaotic carrier from a loss-modulated solid-
state laser, and the first experimental demonstration of chaotic communication with an
optical system (an erbium-doped fibre ring laser) was done in 1998 by Van Wiggeren
and Roy [16–18]. In the past decade research from various groups, both theoretical and
experimental, has focused on understanding synchronization phenomena in unidirectionally
coupled lasers, its potential for use in secure communications, and its dependence on various
laser parameters [19–93].

Synchronization phenomena in spatially extended systems (such as broad-area nonlinear
optical cavities, laser arrays, 1D chains and 2D lattices of coupled dynamical systems, neural
networks, etc.), have also received a lot of attention. Local and global coupling have been considered and two different synchronization regimes have been found [8]. Either all elements of the ensemble display the same behaviour for any initial conditions (full synchronization) or the ensemble splits into groups of mutually synchronized elements (cluster synchronization). In the field of lasers, White and Moloney [94] proposed a practical application of synchronization of spatio-temporal chaos. They demonstrated the successful multiplexing of random bit sequences between transmitter/receiver semiconductor lasers modelled by coupled nonlinear partial differential equations. They reported the synchronization of spatiotemporal chaos through a single scalar complex variable (the total electric field). The separate messages were encoded into individual channels (the longitudinal modes) by injecting weak signals at the relevant mode frequencies, thus enabling the transmission of multiple messages through a single channel. Another application of spatiotemporal chaos synchronization was proposed by García-Ojalvo and Roy [95, 96], who suggested a model system (a broad-area nonlinear optical ring cavity that exhibits spatiotemporal chaos) that allows parallelism of information transfer using an optical chaotic carrier waveform. The synchronization of laser arrays has also been the object of extensive investigation [97–101].

In this chapter we review synchronization of chaotic semiconductor lasers and discuss various types of synchronization in both, unidirectionally and mutually coupled lasers. In Section 6.2 we present numerical and experimental results of synchronization on unidirectional or master–slave configuration. In Section 6.3 we discuss results of the synchronization of mutually coupled semiconductor lasers. In Section 6.4 we present our conclusions and personal perspectives on the research field, at present and in the near future.

6.2 SYNCHRONIZATION OF UNIDIRECTIONALLY COUPLED SEMICONDUCTOR LASERS

Two lasers are coupled unidirectionally if the dynamics of one laser (called master or transmitter) influences the dynamics of the other (called slave or receiver), and the master laser is isolated from the slave laser, so that the dynamics of the slave does not affect the dynamics of the master. A typical experimental setup for unidirectionally coupled external-cavity lasers is shown in Figure 6.1.

Mirasso, Colet and García-Fernández [20] were the first to show theoretically that chaotic semiconductor lasers with optical feedback, unidirectionally coupled, can be synchronized and used in encoded communications systems. In their scheme a message was encoded in the chaotic output of the master laser, and it was transmitted to the slave laser using an optical fibre. The slave laser was assumed to operate under similar conditions as the master laser. The message could be decoded by comparing the chaotic input and output of the slave laser.

Since then, a lot of theoretical and experimental studies have focused on understanding the synchronization properties of unidirectionally coupled chaotic semiconductor lasers [21, 24, 25, 30–32, 34, 36–38, 40–48, 50, 52, 53, 57–61, 63–70, 72, 74–80, 82–86, 88–93]. Synchronization of power dropouts in the low-frequency fluctuations regime
Figure 6.1 Experimental set-up used in [36]: ML – Master Laser; SL – Slave Laser; BS1, BS2, BS3, BS4 – Beam Splitters; PD1, PD2 – Photo Detectors; OI – Optical Isolators; M1, M2 – Mirrors; NDF – Neutral Density Filter; CA – Coupling Attenuator; CRO – Digital Oscilloscope.

(LFF, see Chapter 2) was first shown numerically by Ahlers et al. [34]. The theoretical studies were based on simple rate equations for the complex electric fields and carrier densities in the lasers, which are extensions of the well-known Lang–Kobayashi equations [102] (see Chapter 2). These equations are:

\[ E_m = \frac{1 + i\alpha}{2} \left[ G_m(t) - \frac{1}{\tau_{p,m}} \right] E_m(t) + \kappa_m E_m(t-\tau) \exp(-i\omega_m \tau) + \sqrt{D_m} \xi_m(t) \]  

\[ N_m = \frac{J_m}{e} - \frac{N_m(t)}{\tau_{n,m}} - G_m(t) |E_m(t)|^2 \]  

\[ E_s = \frac{1 + i\alpha}{2} \left[ G_s(t) - \frac{1}{\tau_{p,s}} \right] E_s(t) + \kappa_s E_s(t-\tau) \exp(-i\omega_s \tau) + \eta E_m(t-\tau) \exp[-i(\omega_m \tau_s + \Delta \omega t)] + \sqrt{D_s} \xi_s(t) \]  

\[ N_s = \frac{J_s}{e} - \frac{N_s(t)}{\tau_{n,s}} - G_s(t) |E_s(t)|^2 \]
Here, the indices \( m \) and \( s \) refer to the master and slave lasers respectively. \( E_m, E_s \) are the slowly varying complex fields, and \( N_m, N_s \) are the normalized carrier densities. The equations are written in the reference frame where the complex optical fields of the lasers are given by \( E_m(t) \exp(i \omega_m t), E_s(t) \exp(i \omega_s t) \), where \( \omega_m, \omega_s \) are the optical frequencies of the solitary lasers. The external optical feedback in each laser is described by the parameters \( \kappa_m \) and \( \kappa_s \), which are the feedback levels of the master and slave lasers respectively and \( \tau \) is the delay time in the external cavity. The term \( \eta E_m(t - \tau) \exp[-i (\omega_m \tau + \Delta \omega t)] \) in the right-hand-side of Equation (6.3) accounts for the light injected from the master laser to the slave laser, \( \eta \) is the injection rate, \( \tau \) is the time for light to travel from the master laser to the slave laser and \( \Delta \omega = \omega_m - \omega_s \) is the frequency detuning between the lasers. \( \xi_m(t) \) and \( \xi_s(t) \) are independent complex Gaussian white noises that represent the effects of spontaneous emission and spontaneous recombination. \( D \) measures the noise intensity. The other parameters are as follows: \( \tau_p \) is the photon lifetime, \( \alpha \) is the linewidth enhancement factor, \( G = N/\left(1 + \exp[i/\nu/\omega c]\right) \) is the optical gain, \( \nu \) is the gain saturation coefficient, \( \Delta \omega \) is the phase accumulation after one round-trip in the external cavity, \( J \) is the injection current density, \( e \) is the electronic charge and \( \tau_n \) is the carrier lifetime.

This model does not include multiple reflections in the external cavity, and therefore it is valid for weak feedback levels. It is assumed that the mirrors are positioned such that the external cavity length is the same for both lasers. It is also assumed that the optical field does not experience any distortion during its propagation from the master to the slave laser.

In experiments of synchronization of chaotic external-cavity lasers, two configurations for the slave laser have been considered. In the first configuration, the slave laser is subjected to both, optical feedback from an external reflector and optical injection from the master laser (we will refer to this as \textit{closed-loop configuration}). In the second configuration, the slave laser is subjected only to optical injection from the master laser (we will refer to this as \textit{open-loop configuration}). In the theoretical model, the open-loop configuration corresponds to setting \( \kappa_s = 0 \).

Sivaprasakam and Shore [37] were the first to demonstrate experimentally the synchronization of two chaotic external-cavity semiconductor lasers. The experimental arrangement is shown schematically in Figure 6.1. Two laser diodes (APL-830 of linewidth 200 MHZ) were driven by ultra-low noise current sources (ILX-Lightwave LDX-3620) and their temperature was controlled using thermo-electric controllers (ILX-Lightwave LDT-5412) to a precision of 0.01 K. Both lasers were subjected to optical feedback from external mirrors (M1 and M2) and the feedback strength was controlled using a continuously variable neutral density filter (NDF1). The length of the external cavity is 25 cm in both the cases. The optical isolators (OFR-IO-5-NIR-HP) ensure the lasers were free from unwanted back reflection. The typical isolation is \(-41 \text{ dB}\). Isolator (OI1) ensured the master laser was isolated from the slave laser. The coupling attenuator (CA) enables the percentage of master power fed into slave laser to be controlled. PD1 and PD2 were two identical fast photodetectors (EG&G - FFD040B) with a response time of 2.5 ns. The output of the master laser is coupled to the photodetector (PD1) by the beamsplitters BS1 and BS2. Beam splitter (BS3) acts as the coupling element between the master and slave. Beam splitter (BS4) couples the slave output to the photodetector (PD2). The photodetector output signals are stored in a digital storage oscilloscope (Fluke Combscope PM3394B, 200MHz) and then acquired by a PC.

The master laser and slave lasers were driven into chaos by application of appropriate optical feedback from the external cavity mirrors. The optical feedback for the master and
slave led to a reduction in their threshold currents of 7.0% and 9.7% respectively. (The free-running threshold current ($I_{th}$) values were 57 mA and 53 mA respectively for the master and slave laser.) The temperature and current were adjusted to ensure that the master ($26.66^\circ C, 1.136I_{th}$) and slave ($27.92^\circ C, 1.035I_{th}$) operate at the same wavelength. A small percentage (3%) of the master optical output is injected into the slave and this led to synchronization of the slave to the master. In Figure 6.2(a), the time evolution of the master and slave lasers is shown. The upper trace (master laser) is shifted vertically for clarity. The slave laser actually lags behind the master laser by a time equal to the time for light to travel between the two lasers $\tau_c$, but this cannot be noticed in Figure 6.2(a) due to the different time scales: $\tau_c$ is of the order of nanoseconds, and the time-resolution is of the order of microseconds. The synchronization can conveniently be illustrated by using a synchronization diagram, which consists of a plot of the master intensity vs. the slave intensity. If the two lasers were perfectly synchronized, the synchronization diagram will be a straight line with a positive gradient.

![Figure 6.2](image)

**Figure 6.2** (a) Time traces of the master (upper) and slave (lower) laser output, the time traces are shifted vertically for clarity, (b) the corresponding synchronization diagram [37], (c) Synchronized LFFs for master (ML) and slave (SL) lasers, (d) one shot of enlarged LFF waveforms corresponding to (c) [41].
Figure 6.2 (continued)
Less than perfect synchronization leads to a broadening of the diagram. Figure 6.2(b) shows the synchronization diagram corresponding to Figure 6.2(a).

The frequency detuning between the master and slave lasers is a critical parameter affecting synchronization. The frequency detuning is defined as the master laser frequency minus the slave laser frequency, \( \Delta \omega = \omega_m - \omega_s \). In the experiments of [104] the detuning was varied by modifying the slave laser bias current (a change in the bias current by 1 mA shifts the frequency by 1 GHz, and positive or negative detuning arise depending upon whether the slave laser bias current was increased or decreased). Synchronization plots for four different values of detuning are shown in Figure 6.11. Figure 6.11(a) shows degradation in synchronization for a positive detuning \( (\Delta \omega = 6 \text{ GHz}) \). Good synchronization is shown in Figure 6.11(b), where the gradient is positive. As the detuning is made negative, the synchronization plot starts to branch with a portion of negative gradient as shown in Figure 6.11(c) \( (\Delta \omega = -3 \text{ GHz}) \). As the detuning is further increased to \(-6 \text{ GHz}\) the newly developed branch with a negative gradient dominates and the branch with a positive gradient disappears. This is shown in Figure 6.11(d). The appearance of a negative gradient in the synchronization diagram was termed ‘inverse synchronization’. The output intensities of the lasers, for two different detunings (corresponding to normal and inverse synchronization), are shown in Figure 6.12. These time traces serve as direct time-domain evidence of the inverse behaviour seen in synchronization diagrams of Figure 6.12.

Inverse synchronization was modelled theoretically in [104] based on non-resonant coupling between the master and slave lasers. It was argued that when the injected light is detuned from the slave laser mode wavelength, the coupling into the slave lasing mode is very weak and, therefore, the effects of non-resonant amplification dominate. Even though the injected light is non-resonant with the slave laser cavity mode, it is still amplified through the stimulated emission process. The effect of the non-resonant amplification of light (that does not couple into the slave laser mode) is a reduction of the carriers in the device. Thus, fewer carriers take part in the gain process for the slave laser, reducing its optical output. Wedekind and Parlitz [105] also observed the regime of inverse synchronization (they refer to it as anti-synchronization) and proposed a different theoretical explanation, based on polarization effects due to the Faraday isolator used for the unidirectional coupling. Thus, more theoretical and experimental work is needed in order to clarify the physical mechanisms underlying inverse synchronization phenomena.

The experimental synchronization of LIFF power dropouts was later done by Ohtsubo and co-workers [41] and the time traces are shown in Figure 6.2. In Figure 6.2(c) the upper and lower time trace correspond to the master and slave laser output respectively. The enlarged waveforms of laser outputs are shown in Figure 6.2(d).

From a theoretical point of view the synchronization of two identical lasers in a closed-loop configuration leads to two qualitatively different synchronization regimes [66, 74, 77, 86]. One regime occurs when the master and slave lasers are subjected to the same optical feedback strength:

\[
\kappa_m = \kappa_s,
\]

and the slave laser is subjected to strong enough optical injection (roughly speaking, an injection strength corresponding to the injection locking region of a laser under CW
optical injection). In this regime the slave laser output at time \( t_s \) synchronizes with the injected field at time \( t \), which, taking into account the travel time from the master to the slave, \( \tau_s \), is the field of the master laser at time \( t - \tau_s \). The intensities are related by:

\[
I_s(t) = a I_m(t - \Delta t)
\]  

where \( \Delta t = \tau_s \) and \( a \) is a proportionality constant. Analytic conditions for the occurrence of this type of synchronization were given by Revuelta et al. [75].

A different regime (found numerically by Ahlers et al. [34]) occurs when: (1) the master and slave lasers have the same amount of total external injection, and (2) the master injection strength is strong enough. Condition (1) implies that the master feedback strength is equal to the sum of the slave feedback strength and the optical coupling strength:

\[
\kappa_m = \kappa_s + \eta
\]  

where:

\[
\Delta t = \tau_s - \tau
\]  

A qualitative experimental verification of the condition (6.7) was done in [50]. The output power levels of the master and slave lasers at different current levels were measured when the lasers were synchronized in their output intensity. The feedback power to the transmitter (\( P_{fm} \)) and receiver (\( P_{fr} \)) lasers and the transmitter power coupled to the receiver laser (\( P_c \)) were measured. The quantities \( P_{fm} \), \( P_{fr} \), and \( P_c \) are proportional to \( \kappa_m \), \( \kappa_s \), and \( \eta \) respectively.

Figure 6.4 displays numerical solutions corresponding to anticipated synchronization [55]. In Figures 6.4(a)–(c), the master laser operates in the LFF regime: the intensity suddenly drops toward zero and then recovers gradually, only to drop out again after a random time interval. The intensity dropouts are actually the envelope of a series of fast, picosecond pulses. In Figure 6.4(a) the laser parameters are identical, and the noise level is zero. For large enough coupling strength, \( \eta \), and for an adequate slave feedback level (\( \kappa_s = \kappa_m - \eta \)), the slave laser anticipates by \( \tau - \tau_s \) (5 ns) the output of the master laser. The dotted line
Figure 6.3 Synchronization plots for transmitter-receiver detuning of (a) +6 GHz, (b) 0 GHz, (c) −3 GHz and (d) −6 GHz and with the transmitter laser operating at a low-frequency fluctuation regime [62].

The physical origin of this behaviour can be understood by looking at the simultaneous turn-on of the master and slave lasers, Figure 6.4(b). The lasers emit the first intensity pulse at approximately the same time (because they are identical, and the initial conditions are that both lasers are off at \( t = 0 \)). The master laser emits a train of pulses at the relaxation-oscillation period, before relaxing to the solitary steady state. This train of pulses interferes with the steady state, when it returns from the external mirror, at time \( \tau \) after the emission of the first pulse. A fraction of the master intensity is transmitted to the slave laser, and the train of pulses interferes with the slave laser emission, at time \( \tau_c \) after the emission of the first pulse. Therefore, if the coupling is strong enough, the slave laser will respond in
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Figure 6.4 Time traces of the master (upper traces) and slave (lower traces) lasers for detunings (a) −6 GHz and (b) 0 GHz. The master laser operates in the low-frequency fluctuation regime [62].

A similar manner as the master laser, only that it will do it at time \( \tau_c \), while the master laser will do it at time \( \tau \). The simultaneous turn-on of the lasers allows an understanding of the mechanism of anticipated synchronization, but the lasers also synchronize if they are turned-on independently. In Figure 6.4(c) there are small parameter mismatches between the lasers, and a small amount of noise. The lasers are not perfectly synchronized, and because \( J_s \) is slightly lower than \( J_m - I_s / \tau_c \), fluctuates about a mean value different from zero. Bursts of desynchronization are observed when \( I_m \) drops to zero. Figure 6.4(d) displays solutions corresponding to anticipated synchronization when the master laser operates in the coherence collapse regime (which occurs for larger injection current, see Chapter 2).

In [55] the degree of synchronization and the lag time between the lasers were quantified with the similarity function, defined as:

\[
S^2(\tau_0) = \frac{\langle (I_m(t + \tau_0) - I(t))^2 \rangle}{\sqrt{\langle I_m^2 \rangle \langle I_s^2 \rangle}}
\]

(6.10)

If \( I_m(t) \) and \( I_s(t) \) are independent time series, \( S(\tau_0) \neq 0 \) for all \( \tau_0 \). If the lasers are synchronized such that \( I_s(t) = I_m(t + \tau - \tau_c) \), \( S(\tau - \tau_0) = 0 \). Figure 6.5(a) shows the similarity function when there is perfect anticipated synchronization. \( S(\tau_0) \) presents a sharp minimum at \( \tau_0 = \tau - \tau_c \). There are also additional minima at \( \tau_0 = n(\tau - \tau_c) \) (with \( n \) integer), which arise from
Figure 6.5 $P_{fm}$ vs $(P_{fs} + P_{c})$ under conditions of synchronization [50]. $P_{fm}$ is the feedback power to the master laser, $P_{fs}$ is the feedback power to the slave laser and $P_{c}$ is the power coupled to the slave laser. The size of the squares correspond to a power variation of about 0.01 mW. Maintenance of the synchronization requires current fluctuations less than 0.25 mA, which would produce a change of approximately 0.01 mW in the feedback power.

The dependence of the lag time with the delay time given by Equation (6.9) was verified experimentally by Liu et al. [82]. The experimental arrangement used is shown in Figure 6.6(a). The laser diodes are two similar single-mode DFB laser diodes (NEL-NLK1555) driven with a low-noise high precision injection current, and temperature controlled. At the injection current $I_b = 11.4$ mA (1.5 $I_{th}$) the laser wavelength is 1537.17 nm with a linewidth of about 4 MHz. The intensity variations of the laser outputs are detected by 6 GHz bandwidth photodetectors (New Focus 1514LF) and observed on digital oscilloscope (Tektronix TDS694C) with a 3 GHz bandwidth and a 10 Gps (giga bits per second) sampling rate, as well as on a RF spectrum analyzer (Advantest R#267) with an 8 GHz bandwidth. The light output from the right facet (high-reflection (HR) coated, reflection > 95%) of the master laser is fed back to the left facet (antireflection (AR) coated, reflection < 1%) of the master laser. The light output from the AR-coated facet of the master laser output is injected into the slave laser on the AR-coated facet so that a strong injection can be achieved. The effect of multiple reflections is completely avoided in the current experiment due to the unidirectional ring configuration. The injection power of the slave laser is carefully tuned to match the feedback power of the master laser. The time lag $\Delta t$ between the master and slave laser outputs is calculated from two time series. When the delay time $\tau$ is longer than the propagation time, time lag becomes negative which means the slave laser output anticipates the master laser. Figure 6.6(b) shows the variation of the lag time $\Delta t$ as a function of the delay time $\tau$. Hence, Liu et al. demonstrated anticipated synchronization using the configuration shown in Figure 6.6(b).
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Figure 6.6 Numerical solutions of Equations (6.1)–(6.4) showing anticipated synchronization [55]. The intensities of the master and slave lasers, $I_1$ and $I_2$, have been displaced vertically for clarity: $I_1$ ($I_2$) is the upper (lower) trace. The dotted line indicates the value of $I_1(t) + \tau - \tau c$ $- I_2(t)$. $	au = 10$ ns, and $\tau c = 5$ ns. (a) The lasers are identical and the noise level is zero. The parameters are: $k = 500$ ns$^{-1}$, $\varepsilon = 0$, $\tau c = 1$ ns, $\alpha = 3$, $j = 1.01$, $\omega \tau = 3$ rad, $\gamma_m = 10$ ns$^{-1}$, $\gamma_s = 5$ ns$^{-1}$, $\eta = 5$ ns$^{-1}$. (b) Simultaneous turn-on when the lasers are identical. The parameters are the same as in (a), (c) Same as (a), but the lasers have slightly different injection currents and optical frequencies $(j_i = 1.01 \omega, \omega = 0.299$ GHz), and the noise level is $D_m = D_s = 1 \times 10^{-4}$ns$^{-1}$, $\gamma_i = 0$, $\eta = 10$ ns$^{-1}$, all other parameters as (a). (d) Anticipated synchronization for larger injection current. The lasers are identical and the noise level is zero. $j = 2$, $\gamma_m = 2.5$ ns$^{-1}$, $\gamma_s = 0.25$ns$^{-1}$, $\eta = 2.25$ns$^{-1}$, $\varepsilon = 0.001$, all other parameters as (a).

Synchronization with a time lag $\Delta t = \tau_i$ (Equation 6.6) corresponds to the synchronization of the slave optical field with the injected field and has been referred to as isochronous or generalized synchronization [86]. Synchronization with a time lag $\Delta \tau = \tau_i - \tau$ (Equation 6.8) corresponds to the case when the field of the slave laser anticipates the injected field by an anticipation time equal to $\tau$ and it has been referred to as anticipated or complete synchronization [86].
A numerical comparison of anticipated and isochronous synchronization was done in [74, 75]. It was shown that the two regimes exhibit different sensitivity to spontaneous emission noise, to parameter mismatches and they differ in the response of the slave laser to current modulation of the master laser. In the regime of isochronous synchronization a parameter region exists (for large coupling) so that there is good synchronization even when noise and frequency detuning (of several gigahertz) are taken into account. In contrast, the regime of complete synchronization is easily destroyed by frequency detuning and noise. These results suggest that isochronous synchronization is a form of injection locking, where the slave intensity oscillations are phase locked to the master intensity oscillations, while the amplitudes of the oscillations remain functionally related.

The quality of the synchronization was measured with the correlation coefficients:

\[ C_1 = \frac{\left\{ \left[ I_m(t+\tau_1) - \langle I_m \rangle \right] \langle I_s(t) - \langle I_s \rangle \rangle \right\}}{\left\{ \left[ I_m(t) - \langle I_m \rangle \right]^2 \left\{ I_s(t) - \langle I_s \rangle \right\} \right\}^{1/2}} \]  
(6.11)

and

\[ C_2 = \frac{\left\{ \left[ I_m(t+\tau_2) - \langle I_m \rangle \right] \langle I_s(t) - \langle I_s \rangle \rangle \right\}}{\left\{ \left[ I_m(t) - \langle I_m \rangle \right]^2 \left\{ I_s(t) - \langle I_s \rangle \right\} \right\}^{1/2}} \]  
(6.12)

where \( \tau_1 = -\tau_c \) and \( \tau_2 = \tau - \tau_c \). A large value of \( C_1 \) implies good isochronous synchronization, while a large value of \( C_2 \) implies good anticipated synchronization.

The different sensitivity of the two regimes to frequency detuning is illustrated in Figures 6.7 and 6.8. Figure 6.7(a) displays the synchronization region when the lasers are

![Figure 6.7](image1)

**Figure 6.7** Similarity function (a) for the same parameters as Figure 6.6(a); (b) for the same parameters as Figure 6.6(c). \( S(\tau_o) \) was calculated averaging over 100 time series with different noise realizations [55].
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Figure 6.8. (a) Experimental setup used in [84] to show complete synchronization. LD1 – master laser, LD2 – slave laser, PD – photodiode, OI – Optical isolator, BS – Beamsplitter, HWP – Half-wave plate and NDF – neutral density filter. \( P_{\text{inj}} \) \( (P_{\text{ext}}) \) injection (feedback) power into the receiver (transmitter) laser. (b) Experimentally measured time lag as a function of time delay.

subject to the same feedback level (for large enough coupling isochronous synchronization occurs). The feedback level and bias current are such that both (identical) lasers operate in the regime of coherence collapse. The horizontal axis is the frequency detuning between the lasers, the vertical axis is the optical injection rate, and the grey levels represent the value of \( C_1 \) (the dark grey represents large correlation). The synchronization region is asymmetric and
broad, allowing for frequency detunings up to tens of gigahertz. Figure 6.7(b) displays the same correlation coefficient as Figure 6.7(a) (and for the same parameter values), but when the slave laser is subjected to CW optical injection. The shapes of the chaotic synchronization region in Figure 6.7(a) and the CW injection-locking region in Figure 6.7(b) are very similar, thus suggesting that isochronous synchronization is an injection-locking-type phenomenon. It can also be observed that the chaotic synchronization region is broader than the CW injection-locking region.

Figures 6.8(a) and 6.8(b) display the same as in Figures 6.7(a) and 6.7(b) but in the case of an open-loop configuration ($k_s = 0$). The horizontal axis is the frequency detuning between the lasers, the vertical axis is the optical injection rate, and the grey levels represent the value of $C_1$. Again it can be observed that the chaotic synchronization region in Figure 6.8(a) and the CW injection-locking-region in Figure 6.8(b) are very similar, suggesting that isochronous synchronization is an injection-locking-type phenomenon. However, Fisher et al. [48] experimentally found that two semiconductor lasers in an open-loop unidirectional coupling scheme exhibit chaos-pass-filtering properties: a small perturbation superimposed on the chaotic output of the master laser is filtered out by the slave laser, which is selectively synchronized only to the chaotic dynamics. Thus, this contrasts with the interpretation of isochronous synchronization as an injection-locking-type phenomenon, since in that case the slave laser would amplify both, the chaotic dynamics and the external perturbation in the same way.

Comparing Figures 6.7(a) and 6.8(a) (that are done with the same grey-scale), it is clear that the synchronization quality is in general lower when the slave laser does not have its own external optical feedback. A possible explanation to this is based on the fact that, when the master and slave lasers are both external cavity lasers subjected to the same amount of optical feedback, under appropriate conditions an analytical synchronized solution exists in which the field of the slave laser lags by $\tau_c$ the field of the master laser [76]. When the slave laser is a solitary laser, no such solution exists.

Thus, the synchronization with a close-loop scheme has advantages for applications where a high degree of synchronization is required. However, it has the disadvantage that additional components have to be used in the experimental set-up. In particular, the external mirror at the slave laser has to be very carefully positioned because small differences in the external cavity feedback phases ($\omega_m \tau_c - \omega_s \tau_c$) can strongly degrade the synchronization quality. Peil et al. [79] found experimentally and numerically a rich and complex set of outputs when varying the relative optical feedback phase. Their results demonstrate a characteristic synchronization scenario in dependence on $\omega_m \tau_c - \omega_s \tau_c$, leading cyclically from chaos synchronization to intermittent synchronization to almost uncorrelated oscillations, and back to chaos synchronization.

In the case of two identical lasers in an open-loop scheme, anticipating synchronization occurs for zero detuning and $\eta = \kappa_m$. Thus, it is interesting to study the value of $C_2$ in the parameter space (detuning, injection rate). This is displayed in Figure 6.8(c), which shows $C_2$ on the same grey-scale as Figures 6.7(a) and 6.8(a). Comparing Figures 6.8(a) and 6.8(c) it is clear that the correlation coefficient $C_2$ is usually larger than the correlation coefficient $C_1$. However, in the region $\eta \sim \kappa_m$ and small detuning, $C_2 > C_1$. It can be observed that the dark region where good anticipating synchronization occurs is narrower than in the isochronous case.

The two regimes of synchronization also exhibit different sensitivity to the noise level. This sensitivity is illustrated in Figure 6.9, which displays the synchronization diagrams: master
output intensity, $I_m$, vs. slave output intensity, $I_s$. In Figure 6.9(a) the feedback levels are $\kappa_s = \kappa_m$ and $I_m$ is shifted in time by $-\tau_c$; in Figure 6.9(b) the feedback and injection levels are related by $\eta + \kappa_s = \kappa_m$ and $I_m$ is shifted in time by $\tau - \tau_c$. The noise level in both cases is the same, but the points in Figure 6.9(a) are clearly concentrated along a straight line, whereas in Figure 6.9(b) they are more scattered owing to noise-induced large bursts of desynchronization.

It has been shown numerically that by varying certain parameters a transition from anticipated to isochronous synchronization can be observed [77, 86]. An example of such transition is illustrated in Figure 6.10, which shows the master laser and slave laser intensities (averaged in time to simulate the typical bandwidth of the detectors used in experiments). Figure 6.10(a) displays $I_m(t-\tau_c)$, while Figures 6.10(b), 6.10(c) and 6.10(d) display $I_s(t)$ for different injection rates, $\eta$. In Figure 6.10(b) $\eta = \kappa_m$ and $I_s(t)$ anticipates the injected intensity $I_m(t-\tau_c)$ by an anticipation time $\tau (\approx 1$ ns$)$. In Figure 6.10(c) $\eta$ is slightly larger.

**Figure 6.9** Comparison between isochronous synchronization and CW injection-locking when the slave laser is an external-cavity laser [88]. (a) Correlation coefficient $C_1$ as a function of the frequency detuning and the injection rate when the slave is an external cavity laser subjected to chaotic injection from the master laser. $\gamma_m = \gamma_s = 10$ ns$^{-1}$. (b) Correlation coefficient $C_1$ as a function of the frequency detuning and the optical coupling strength, when the slave is an external cavity laser subjected to CW injection from the master laser. $\gamma_s = 10$ ns$^{-1}$. 
than $\kappa_m$ and isochronous synchronization occurs. The time traces shown in Figures 6.10(a) and 6.10(c) are most of the time equal, the main difference being a less pronounced dropout in the intensity of the slave laser. In Figure 6.10(d) $\eta$ is slightly less than $\kappa_m$ and synchronization disappears: the time traces shown in Figures 6.10(a) and 6.10(d) are completely different.

Unidirectionally coupled external cavity semiconductor lasers operating on the LFF regime can also exhibit intermittent synchronization. Wallace et al. [52] have shown that the outputs of master and slave lasers are synchronized in their sudden dropouts, but remain uncorrelated in the process of power recovery. This is because during the recovery process the intensity of the master laser is very low and the optical injection from the master to the slave laser is not strong enough to govern the dynamics of the slave laser.

Synchronizing several lasers becomes important when the lasers are used in practical communication systems. Sivaprakasam and Shore experimentally demonstrated the possibility of synchronizing three chaotic external cavity diode lasers in a cascade scheme [72]. The experimental set-up is shown in Figure 6.13. The master laser and both the slave lasers are driven to LFF by application of appropriate feedback from the external cavity. The effective external reflectivities for the master, slave-1 and slave-2 are $1.6 \times 10^{-3}$, $5.0 \times 10^{-4}$, and $4.0 \times 10^{-3}$ respectively. The free running threshold current ($I_{th}$) values are 56 mA, 49 mA and 55 mA respectively for master, slave-1 and slave-2 laser. The

![Figure 6.9 (continued)](image)
temperature and current are adjusted to ensure that the master (25.66 °C, 1.17I_m) and slave-1 (26.35 °C, 1.20I_m) operate at the same wavelength. A small percentage (15%) of the master optical output power is fed to the slave-1, which leads to the synchronization of slave-1 to the master. 6.5 percentage of the slave-1 output power is fed to the slave-2 laser. The temperature and operating current of the slave-2 (26.96 °C, 1.15I_m) are adjusted so as to obtain synchronization between the slave-1 and slave-2 lasers. The process of synchronizing slave-2 and slave-1 laser does not affect the synchronization between the slave-1 and master laser. Hence, all the three lasers are synchronized.

The time evolution of the intensities of the three lasers is shown in Figure 6.14: trace (a), (b) and (c) are the output intensities of master, slave (1) and slave (2) lasers respectively. It is noticeable from this figure that the output of slave laser (1) lags behind master laser, and slave laser (2) lags behind the slave laser (1).

Figure 6.10 Comparison between isochronous synchronization, CW injection-locking, and anticipated synchronization when the slave laser is a solitary laser (\(\gamma_s = 0\)) [88]. (a) Correlation coefficient \(C_1\) as a function of the frequency detuning and the injection rate, when the slave laser is subjected to chaotic injection from the master laser. \(\gamma_m = 10 \ \text{ns}^{-1}\). (b) Correlation coefficient \(C_1\) as a function of the frequency detuning and the optical coupling strength, when the slave laser is subjected to CW injection from the master laser. (c) Correlation coefficient \(C_2\) as a function of the frequency detuning and the optical coupling strength, when the slave laser is subjected to chaotic injection from the master laser.
Figure 6.10 (continued)
Figure 6.11 Effect of noise in the isochronous and anticipated synchronization regimes [76]. (a) $I_m(t - \tau_c)$ is plotted against $I_s(t)$. The parameters are $\gamma_m = \gamma_s = 20 \text{ ns}^{-1}$, $\eta = 50 \text{ ns}^{-1}$, $\Delta \omega = 0$, $D = 0.02 \text{ ns}^{-1}$, $C_1 = 0.99$ (b) $I_m(t + \tau - \tau_c)$ is plotted against $I_s(t)$. The parameters are $\gamma_m = 20 \text{ ns}^{-1}$, $\gamma_s = 0$, $\eta = 20 \text{ ns}^{-1}$, $\Delta \omega = 0$, $D = 0.02 \text{ ns}^{-1}$. $C_2 = 0.94$.

Figure 6.12 Transitions between anticipated and isochronous synchronization are illustrated by plotting the time traces of the time-averaged master and slave intensities [88]. (a) Intensity of the master laser (lagged $\tau_c$ in time) for $\gamma_m = 10 \text{ ns}^{-1}$. Intensity of the slave laser for (b) $\eta = \gamma_m$; (c) $\eta = 12 \text{ ns}^{-1}$; (d) $\eta = 9 \text{ ns}^{-1}$. 
Figure 6.13 Schematic diagram of the experiment of cascade synchronization in Ref. [74]: ML – Master laser; SL1 – Slave laser-1; SL2 – Slave laser-2; M1– M3 – external cavity mirrors; NDF’s – neutral density filters, BS1–BS7 – Beam splitters, PD1–PD3 – Photodetectors, OI1–OI5 – Optical isolators, CA1 and CA2 – Coupling attenuator.

6.3 SYNCHRONIZATION OF MUTUALLY COUPLED SEMICONDUCTOR LASERS

A lot of research has focused on mutually coupled lasers, either subjected only to mutual injection, or subjected to mutual injection and optical feedback from an external mirror [106–117]. When the lasers are subjected only to their mutual optical injection and have dissimilar intensities, their coupling strength may be asymmetric, and in this case, Hohl et al. [106] found that the coupled lasers exhibit a form of synchronization which is characterized by small oscillations in one laser, and large oscillations in the other. Heil et al. [109] have found that two mutually coupled lasers may exhibit subnanosecond synchronized chaotic dynamics, in which a leading laser synchronizes its lagging counterpart.
Figure 6.14 Cascade synchronization [74]. Time-series plots of (a) master laser, (b) slave laser-1, and (c) slave laser-2. The time traces have been shifted in y-axis for clarity.

The time lag between the intensities is equal to the time for light to travel from one laser to the other. This effect was interpreted as a spontaneous symmetry-breaking phenomenon. Fujino and Ohtsubo also observed the synchronization of the chaotic outputs from mutually injected lasers [111]. It was shown that the synchronization mechanism was not based on complete chaos synchronization but on injection-locking phenomena (which the authors refer to as generalized chaos synchronization). A recent theoretical study on synchronization of mutually coupled multimode semiconductor lasers shows that the two lasers can be synchronized within each laser-cavity mode, while the synchronization across different cavity modes is significantly weaker [112].
The rate equations commonly used in the literature to model mutually coupled single-mode lasers are

\[
\dot{E}_{1,2} = \frac{1 + j\alpha}{2} \left[ G_{1,2}(t) - \frac{1}{\tau_{p,1,2}} \right] E_{1,2}(t) + \kappa_{1,2} E_{2,1}(t - \tau_c) \tag{6.13}
\]

\[
\dot{N}_{1,2} = \frac{J_{1,2}}{e} - \frac{N_{1,2}(t)}{\tau_{n,1,2}} - G_{1,2}(t) |E_{1,2}(t)|^2 \tag{6.14}
\]

where \( E_1 \) and \( E_2 \) are the complex optical fields of the two lasers, \( N_1 \) and \( N_2 \) are the carrier densities in the two lasers, and the other parameters have the same meaning as in the previous section. These equations were derived in [116], in the limit of small mutual coupling, starting from Maxwell equations, supplemented with adequate boundary conditions.

In the experiments of [110] a different mutual coupling configuration was used. The experimental set-up is shown in Figure 6.15. Two single-mode (side mode suppression ratio equal to \(-20\) dB), Fabry–Perot laser diodes emitting at 830 nm were used, but only one is subjected to optical feedback from an external reflector (this laser will be called external-cavity laser, or laser 1, while the laser without external feedback will be called solitary laser, or laser 2). The laser operating temperatures were stabilized using thermo-electric elements.

**Figure 6.15** Experimental setup used in Ref. [110]. ML – external-cavity laser, SL – solitary laser, BS1–3, Beam Splitters, NDF – Neutral density filter, OI1–3, Optical Isolators, M1–2, Mirrors, CA – Coupling Attenuator, PD1–2, Photodetectors, OSA – Optical spectrum analyzer.
Synchronization of Mutually Coupled Semiconductor Lasers

and controllers to a precision of 0.01 K. The output of each laser was coupled to a fast
photodetector (Newport - AD-70xr) and monitored using a digital oscilloscope (LeCroy-
LC564A). The optical isolators ensure that no feedback from the photodetectors reaches the
laser diodes. Laser 1 (laser 2) was biased at 1.08 (1.04) times the free-running threshold.
The time of flight between the two lasers was 3.5 ns. Laser 1 was operated in an external
cavity configuration with external reflectivity $1.75 \times 10^{-3}$, which drives the laser into the
LFF regime (when the external mirror was removed, no power dropouts were observed).
Beamsplitters BS1 and BS3 coupled the transmitter laser output to the receiver laser. The
coupling attenuator (CA) was used to control the amount of light coupled between the lasers
(the percentage of laser 1 output power reaching laser 2 and vice versa is 0.14% throughout
the experiment). It is noted that, when the optical feedback from the external cavity mirror
is not applied, the lasers did not show any LFF dynamics.

Figure 6.16 shows the time traces of the intensities of laser 1 and laser 2. It can be
seen that close to 250 ns and 450 ns the output of laser 2 drops and recovers ahead of the
output of laser 1. Hence, laser 2 is leading laser 1 by an ‘anticipation time’, $\tau_A = 3.5$ ns,
which was measured with the digital oscilloscope. The measurement of anticipation time
was confirmed by studying the quality of synchronization between the laser 1 and 2 outputs.
The laser 2 output was plotted against laser 1 output so as to obtain the synchronization
plot. The synchronization plot was then least square fitted to a straight line and the slope ($m$)
and its variation ($\Delta m$) was calculated. The inverse of the variation ($1/\Delta m$) represents the
quality of the synchronization ($S_Q$). Good synchronization would be indicated by $m = 1$
and low variation ($\Delta m$) implying high synchronization quality. On the other hand, poor
synchronization would give a relatively large variation ($\Delta m$) and hence a low synchronization

![Figure 6.16](image)

**Figure 6.16** Time traces of the external-cavity laser (upper) and solitary laser (lower) outputs [110].
The vertical lines identify that the solitary laser is ahead of the external-cavity laser. The external
cavity round trip time is 13.5 ns.
quality. Figure 6.17 shows the dependence of the (normalized) synchronization quality ($S_Q$) on the laser 1 time shift $\tau_s$, where $\tau_s$ is the time by which the laser 1 output is shifted relative to the laser 2 output. It is shown for two external-cavity delay times. It can be seen that the synchronization quality, in both cases, shows a sharp maximum at $\tau_s = 3.5$ ns, which is the flight time from one laser to the other. This indicates that laser 2 is leading laser 1 by 3.5 ns irrespective of the external-cavity round-trip time. Thus, in mutual coupling a regime of anticipating synchronization was found in which the anticipation time does not depend on the external-cavity round-trip time of the laser 1. Anticipating synchronization depends on the operating wavelength of both lasers. Once the lasers are synchronized, both the lasers emit at a common wavelength denoted by $\lambda_{sys}$, and it is found that $\lambda_{sys} < \lambda_1$ (where $\lambda_1$ is the central operating wavelength of laser 1). The operating temperature of one of the lasers is modified by a fraction of a degree such $\lambda_{sys} < \lambda_1$, and hence rendering laser 2 to lag behind laser 1 [114].

6.4 CONCLUSION

The aim of this chapter was to give an overview of the field of synchronization of chaotic semiconductor lasers, from the theoretical and experimental points of view. In Section 6.2 we discussed synchronization of unidirectionally coupled lasers (in master–slave or driver-response configuration). We considered the case in which the slave laser has its own feedback (closed-loop scheme) and the case in which it is a solitary laser (open-loop scheme). We discussed the two well-known synchronization regimes: isochronous, which has been interpreted as an injection-locking-type phenomenon that requires the frequency detuning between the lasers to be within a certain injection-locking range, and anticipated synchronization, which requires perfect matching of the optical frequencies and all the internal laser parameters. In terms of the well-known synchronization regimes of chaotic systems, anticipated synchronization is identified with complete synchronization, and isochronous synchronization is identified with generalized synchronization. We reviewed
recent, exciting developments such as inverse synchronization, which is a phenomenon not yet fully understood. In Section 6.3 we discussed synchronization of mutually coupled lasers, in a face-to-face configuration, and in the case when one of the lasers is also subjected to its own feedback from an external mirror. We have chosen to discuss only the synchronization of edge-emitting lasers and not of vertical-cavity surface-emitting lasers, because, in spite of the fact that there are several theoretical studies that indicate the possibility of synchronizing VCSELs, no experimental demonstration has been done to the best of our knowledge. We hope that future developments of technology towards secure all-optical communications based on chaotic synchronized systems will clarify relevant issues such as the physical mechanisms underlying complete and generalized synchronizations.

REFERENCES


