3.3 Distributions of Time Spent in System and the Waiting Time Prior to Service in a FCFS M/G/1 Queue

Given the distribution for the number in the system as seen by a departing customer, it is fairly straightforward to derive the distribution of time spent waiting in queue and total time spent in system by a customer. We outline the approach next as this approach is very commonly used in analysing the delays of a queueing-type system. The distribution for a LCFS queue is much more difficult to obtain and will be postponed for Section 3.5

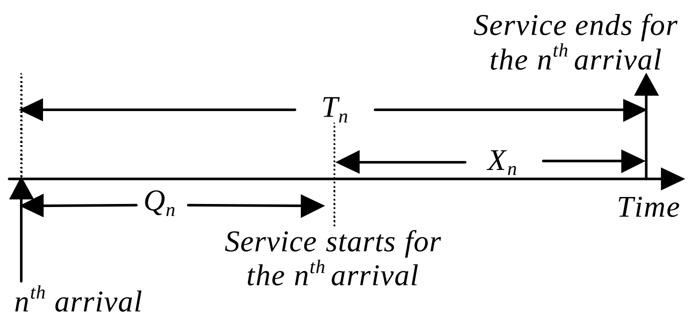

Figure 3.5. Time Instants of Arrival and Departure for a Customer in a FCFS M/G/1 Queue

For a FCFS M/G/1 queue, consider the n^{th} arrival to the queue as shown in Figure 3.5. As shown in the figure, this arrival waits in the queue for a time interval of length Q_n before its service can start. (Note that $Q_n=0$ if the arrival enters an empty queue.) Once service starts, the customer engages the server for a time interval X_n corresponding to its service time. The total time T_n spent in the system by the n^{th} arrival is then Q_n+X_n.

As before, we use $b(t)$ to denote the probability density function of the service time X_n with its L.T. given by $L_B(s)$. Similarly, let $f_Q(t)$ be the probability density function of the queueing delay Q_n with L.T. $L_Q(s)$ and let $f_T(t)$ be the probability density function of the total time T_n spent in system by the n^{th} arrival with L.T. $L_T(s)$. It is important to note here that the probability density function's of these random variables and their L.T.s have been written without explicitly mentioning the variable n (for the n^{th} arrival) since the system is assumed to have reached equilibrium conditions.

Consider the n^{th} arrival as shown in Figure 3.5. Since the queue is assumed to be FCFS in nature, the number of customers that the n^{th} user will see left behind in the queue when it departs will be the number of arrivals that occur while it is in the system. The generating function for this random number (at equilibrium) has already been obtained as *P(z)* in Eq. (3.14) as the generating function of the number in the system as seen by a departing customer.

We can also argue that the generating function *P(z)* for the number in the system when this n^{th} customer departs can also be obtained in another way. Using the approach used to get Eq. (2.36) and Eq. (3.13), we can argue that this will be given as $L_T(l-lz)$ where $L_T(s)$ is the L.T. of the probability density function of the total time *T* spent in the system by an arriving customer. Therefore, we get that

$$L_T(l - lz) = \frac{(1-r)(1-z)L_B(l - lz)}{L_B(l - lz) - z}$$

Substituting $s=(l - lz)$, we then get

$$L_T(s) = \frac{s(1-r)L_B(s)}{s - l + lL_B(s)} \tag{3.15}$$

The L.T. $L_T(s)$ may then be inverted to obtain the probability density function $f_T(t)$ of the total time spent by an arriving customer in a FCFS M/G/1 queue. It is also possible to directly use $L_T(s)$ to obtain the desired moments of the random variable *T*, the total time spent in system by an arriving customer.

Knowing $L_T(s)$, and using the fact that $Q+X=T$, where *Q* and *X* are independent of each other (i.e. Q^X), we get

$$L_Q(s) = \frac{L_T(s)}{L_B(s)} = \frac{s(1-r)}{s - l + lL_B(s)} \tag{3.16}$$

as the L.T. of the probability density function $f_Q(s)$ of the queueing delay *Q*. Inverting this Laplace Transform will give the associated probability density function $f_Q(t)$ of the queueing delay. Even if this inversion is not explicitly done, the moments of *Q* may be found directly from $L_Q(s)$ itself by differentiating it and evaluating the differential at $s=0$ using the moment generating properties of Laplace Transforms.